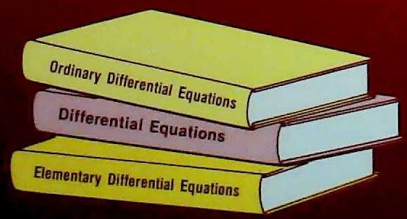
DIFFERENTIAL EQUATIONS

EXAM FILE

27 Professors Reveal Their Examinations



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4. THE LAPLACE TRANSFORM 317
Forward Transform of Elementary Functions 317
Inverse Tranform of Selected Functions 327
Shifting on the S-Axis 334
Heaviside Functions and Shifting on the T-Axis
Dirac's Delta Function. Periodic Functions 344
Differentiation and Integration 359
Convolution 366
Solution of Differential Equations with
Initial Conditions 379
Miscellaneous Results 406

5. APPLICATIONS OF ORDINARY DIFFERENTIAL EQUATIONS 415
Orthogonal Trajectories 415
Growth and Decay 429
Mixing Problems 446
Physical Problems 454
Harmonic Motion 474
Electric Circuits 486
Miscellaneous 494

Contents

Foreword	v
1. FIRST ORDER EQUATIONS	1
Separable	1
Exact	17
Homogeneous	39
Integrating Factors	51
Linear	68
Reduction to First Order	85
Picard	98
Miscellaneous Techniques	103
2. SECOND ORDER LINEAR EQUATIONS	131
Constant Coefficient, Homogeneous	131
Euler Equations, Homogeneous	148
Reduction of Order	163
Solution of Inhomogeneous Equations by use of Undetermined Coefficients	179
The Method of Variation of Parameters	212
Miscellaneous Techniques	236
3. SOLUTION BY POWER SERIES	249
Review of Power Series	249
First Order Equations	255
Classification of the Base Point	262
Second Order with $X = X_0$ an Ordinary Point	271
Second Order with $X = X_0$ a Regular Singular Point (Frobenius Series)	289
Distinct Roots Not Differing by an Integer	289
Equal Roots	309
Distinct Roots Differing by an Integer	315
Miscellaneous Techniques	316

Foreword

Preparing for exams is an important part of any mathematics course. Class notes, textbooks, and textbook supplements have traditionally been the main input sources for this purpose. All too often such sources have been inadequate, since the questions that professors actually ask are not quite the same as those in the sources. This is true for several reasons, including different priorities of the professors, his or her awareness of time constraints on test questions, and others.

Since students of differential equations would like to know on which aspects of the subject they are going to be tested, the kind of questions that will be asked, and what professors think are good answers to those questions, this book is designed to provide that information. Many math professors have looked in their exam files and have contributed their favorite differential equation questions. These questions are given exactly as they appeared in actual course exams, and the solutions are in the professors' own handwriting.

No attempt has been made to impose a uniform format on the problems or the solutions. Thus you will find diverse problem solving styles, notation, and even language. This diversity should prove to be one of the real strengths of this book. Each of the contributing professors has been careful not to make errors, but nobody is perfect and it is possible that an error has escaped their attention. If you find one, a note mailed to Engineering Press will be appreciated.

If, as you study the problems given here, you see some problems that are completely unfamiliar to you, this is because different differential equation courses can emphasize very different aspects of the subject. It should be quickly evident to you which these problems are, and you should treat them as potential enrichments rather than basic study material. There are also wide variations in the level of difficulty among these problems.

We also publish similar books for Calculus I, II, and III as well as for Linear Algebra. We expect that these books will help you do well in the course you are taking. In particular, if you have difficulty with the integrations required in these problems, we recommend Calculus II Exam File, chapter two.

D.R. Arterburn
Editor

1-2

Solve $\frac{dy}{dx} = \sqrt{xy}$ (find the general solution)

****

$$\frac{dy}{dx} = \sqrt{xy} = \sqrt{x}\sqrt{y} = x^{\frac{1}{2}} y^{\frac{1}{2}}$$

$$y^{-\frac{1}{2}}\, dy = x^{\frac{1}{2}}\, dx$$

$$2y^{\frac{1}{2}} = \frac{2}{3} x^{\frac{3}{2}} + k$$

$$y^{\frac{1}{2}} = \frac{1}{3} x^{\frac{3}{2}} + c \quad \text{where } c = \frac{1}{2}k$$

$$y = \left(\frac{1}{3} x^{\frac{3}{2}} + c\right)^2$$

1-3

Find a general solution: $(2-x)y' = y^2$

****

$$\frac{dy}{dx} = \frac{y^2}{2-x}$$

$$y^{-2}dy = \frac{dx}{2-x}$$

$$-\frac{1}{y} + \int \frac{dx}{x-2} = C$$

$$-\frac{1}{y} + \ln|x-2| = C$$

1
FIRST ORDER EQUATIONS

SEPARABLE

1.1

Solve the initial value problem
$(x^2y + xy - y)dx + (x^2y - 2x^2)dy = 0$, $y(-e)=1$.

****

$(x^2y + xy - y)\,dx + (x^2y - 2x^2)\,dy = 0$ separable

$y(x^2+x-1)\,dx + x^2(y-2)\,dy = 0$

$\frac{x^2+x-1}{x^2}\,dx = \frac{-(y-2)}{y}\,dy$

$(1 + \frac{1}{x} - x^{-2})\,dx = (-1 + \frac{2}{y})\,dy$

$x + \ln|x| + \frac{1}{x} + C = -y + 2\ln|y|$

$y(-e) = 1$: $\quad -e + 1 - \frac{1}{e} + C = -1 + 0 \qquad C = e + \frac{1}{e} - 2$

solution of ivp: $x + \ln|x| + \frac{1}{x} + e + \frac{1}{e} - 2 = -y + 2\ln|y|$

1-6

If you borrow money at a rate r, and pay back \$m per month, then P, the amount you owe, is governed by

$$\frac{dP}{dt} = rP - 12\,m \quad (t \text{ in years}).$$

If you borrow \$50,000 at 12% interest, how much must you pay per month to pay off the loan in 20 years?

****

Since we borrow \$50,000, $P(0) = 50{,}000$.
If we pay off the loan in 20 years, $P(20) = 0$. $r = .12$, what is m?

$$\frac{dP}{dt} = .12P - 12m = \frac{12}{100}(P - 100m)$$

Separate variables (m is constant):

$$\frac{dP}{P-100m} = \frac{12}{100}\,dt \qquad \text{Integrate:}$$

$$\ln|P-100m| = \frac{12}{100}t + C \qquad \text{Exponentiate:}$$

$$|P-100m| = e^{\frac{12}{100}t}\cdot e^{C} \Rightarrow$$

$$P-100m = Ke^{\frac{12}{100}t}, \qquad \boxed{P = 100m + Ke^{\frac{12}{100}t}}$$

Now $P(0) = 100m + K = 50{,}000$

and $P(20) = 100m + Ke^{12/5} = 0$

To solve for m multiply the top equation by $e^{12/5}$ and subtract:

1-4

Solve the differential equation $e^{y^2} + x^2 y\, y' = 0$.

****

Use separation of variables to obtain

$$x^2 y\, y' = -e^{y^2}.$$

Thus $$y e^{-y^2} \frac{dy}{dx} = -\frac{1}{x^2}.$$

Or $$y e^{-y^2} dy = -x^{-2} dx.$$

Thus $$\int y e^{-y^2} dy = -\int x^{-2} dx,$$

and we obtain $$-\frac{1}{2} e^{-y^2} = \frac{1}{x} + C.$$

We write the final equation as

$$\frac{1}{e^{y^2}} = -\frac{2}{x} + C.$$

1-5

Solve the initial value problem

$$y' = 3(xy)^2; \quad y(1) = 1$$

****

Separable; $\frac{dy}{dx} = 3x^2 y^2$ gives $\frac{dy}{y^2} = 3x^2 dx$.

Integrating, $-\frac{1}{y} = x^3 + c$. Initial values: $-1 = 1 + c$

$c = -2$ gives $-\frac{1}{y} = x^3 - 2$; $\frac{1}{y} = 2 - x^3$; $y = \frac{1}{2 - x^3}$.

1-8

Solve the initial value problem $y' = (\cos 3x - 1)/y$, $y(0) = -2$ and write y explicitly as a function of x.

****

Rewrite the equation as $\frac{dy}{dx} = \frac{\cos 3x - 1}{y}$: separable.

$y\,dy = (\cos 3x - 1)\,dx$. Integrating,

$\frac{y^2}{2} = \frac{\sin 3x}{3} - x + C$. With $x = 0$ and $y = -2$,

$\frac{4}{2} = \frac{\sin 0}{3} - 0 + C$, or $2 = C$. Therefore,

$\frac{y^2}{2} = \frac{\sin 3x}{3} - x + 2$. Solving for y,

$y^2 = \frac{2\sin 3x}{3} - 2x + 4$, $y = \pm\sqrt{\frac{2\sin 3x}{3} - 2x + 4}$

Since $y(0) = -2$, the minus root is correct :

$y(x) = -\sqrt{\frac{2\sin 3x}{3} - 2x + 4}$.

1-9

Find the solution of $e^{y-x}\,dx + e^{x-y}\,dy = 0$ which passes through the origin.

****

Multiply by e^{-x-y} and obtain the D. E.

$e^{-2x}\,dx + e^{-2y}\,dy = 0$, a separable D. E.

Since the initial condition is $y(0) = 0$, the

$$100\,e^{12/5}\,m + k\,e^{12/5} = 50{,}000\,e^{12/5}$$
$$-\,(100\,m + k\,e^{12/5} = 0\;)$$
$$100\,(e^{12/5}-1)\,m = 50{,}000\,e^{12/5}$$
$$\boxed{m = 500\,\frac{e^{12/5}}{e^{12/5}-1} \cong \$549.\underline{88}}$$

1-7

Upon algebraic simplification, the following first order ordinary differential equation is seen to be 'separable':

$$[2x\cdot dx + (2 - x + 2y)\cdot dy + xy\cdot(dx - dy)] = 0.$$

Determine the general solution to the given differential equation.

****

Expressing the given equation in differential form, we write $(2x+xy)\cdot dx + (2-x+2y-xy)\cdot dy = 0$, or

$$x\cdot(2+y)\cdot dx + [2(1+y) - x\cdot(1+y)]\cdot dy = 0, \text{ or}$$
$$x\cdot(2+y)\cdot dx + (1+y)\cdot(2-x)\cdot dy = 0.$$

Via Separation of Variables, we have

$$\int \frac{x}{2-x}\cdot dx + \int \frac{1+y}{2+y}\cdot dy = 0, \text{ or}$$
$$\int \left[-1+\frac{2}{2-x}\right]\cdot dx + \int \left[1-\frac{1}{2+y}\right]\cdot dy = 0$$

Anti-differentiation then gives the general solution:

$$-x - 2\cdot \ln|2-x| + y - \ln|2+y| = c.$$

1-11

Solve: $(xy + y)y' = x^2y^2 + x^2 + y^2 + 1$ such that when $x = 2$, $y = 0$.

****

WE FACTOR: $(xy+y) = y(x+1)$. $x^2y^2+x^2+y^2+1 = (x^2+1)(y^2+1)$.

THEN $y' = \dfrac{(x^2+1)(y^2+1)}{y(x+1)}$. $y' = \dfrac{dy}{dx}$, SO WE CAN "SEPARATE THE VARIABLES": $\dfrac{y\,dy}{y^2+1} = \dfrac{(x^2+1)dx}{x+1} = \left(x-1+\dfrac{2}{x+1}\right)dx$

INTEGRATING BOTH SIDES: $\frac{1}{2}\ln(y^2+1) = \frac{x^2}{2} - x + 2\ln|x+1| + C$.

THIS IS A GENERAL SOLUTION FOR THE GIVEN DIFFERENTIAL EQUATION.

WHEN $x=2$, $y=0$, SO $\frac{1}{2}\ln(0^2+1) = \frac{2^2}{2} - 2 + 2\ln|2+1| + C$.

SIMPLIFYING AND SOLVING FOR C, $C = -2\ln 3$. (NOTE: $\ln 1 = 0$.)

$\therefore$ $\frac{1}{2}\ln(y^2+1) = \frac{x^2}{2} - x + 2\ln|x+1| - 2\ln 3$. THIS IS AN ACCEPTABLE FORM FOR THE SOLUTION TO THE PROBLEM, BUT IT CAN BE MANIPULATED IN MANY WAYS, e.g.,

$$y^2 = \left(\frac{x+1}{3}\right)^4 e^{x^2-2x} - 1.$$

1-12

Solve the initial value problem

$$y'/\ln y = ye^x; \quad y(0) = e^e.$$

****

This is a separable equation;

$\dfrac{dy}{y\ln y} = e^x dx$. To find $\int \dfrac{dy}{y\ln y}$,

solution is $\int_0^x e^{-2t}dt + \int_0^y e^{-2s}ds = 0$, or

$-\frac{1}{2}(e^{-2x}-1) + -\frac{1}{2}(e^{-2y}-1) = 0$, or

$e^{-2x} + e^{-2y} = 2$ (implicit solution)

$y = -\frac{1}{2}\ln(2-e^{-2x})$ (explicit solution).

1-10

Solve $(xy + x)^2 y' + (xy + y)^2 = 0$

****

This is a separable first order differential equation and we write it as

$$(xy+y)^2 dx + (xy+x)^2 dy = 0$$

$$y^2(x+1)^2 dx + x^2(y+1)^2 dy = 0$$

$$\frac{(x+1)^2}{x^2} dx + \frac{(y+1)^2}{y^2} dy = 0$$

$$\left(\frac{x^2+2x+1}{x^2}\right) dx + \left(\frac{y^2+2y+1}{y^2}\right) dy = 0$$

Therefore, the solution is given by

$$\int\left(1 + \frac{2}{x} + \frac{1}{x^2}\right) dx + \int\left(1 + \frac{2}{y} + \frac{1}{y^2}\right) dy = C$$

where C is an arbitrary constant.
By integration we obtain

$$x + 2\ln|x| - \frac{1}{x} + y + 2\ln|y| - \frac{1}{y} = C$$

that is

$$x + y - \left(\frac{1}{x} + \frac{1}{y}\right) + \ln x^2 y^2 = C$$

1-14

$\frac{dx}{dy} = 2(x^2 + 1)y$

(A) $\tan x + C = y^2$

(B) $\ln(x^2 + 1) = y^2 + C$

(C) $\arctan y^2 = x + C$

(D) $x = \tan(y^2 + C)$

(E) $\arctan x + y^2 = C.$

****

We first separate the variables x and y:

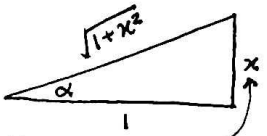
$$2y\,dy = \frac{1}{x^2+1}\,dx$$

Then integrate both sides

$$y^2 + C_0 = \int \frac{1}{x^2+1}\,dx$$

(C_0: constant)

To perform the x-integration, we make a change of variable: Note that $1+x^2$ is the square of the hypothaneous of a right triangle with sides 1 and x



We put the x here so that we have $\tan\alpha = \frac{x}{1}$

(We like $\tan\alpha$ better than $\cot\alpha$; The derivative of $\tan\alpha$ is $\frac{1}{\cos^2\alpha}$, the derivative of $\cot\alpha$ is $\frac{-1}{\sin^2\alpha}$ and we don't want to be bothered with minuses).

Since $\frac{1}{\sqrt{1+x^2}} = \cos\alpha$, we have $\frac{1}{1+x^2} = \cos^2\alpha$

and we replace $\frac{1}{1+x^2}$ by $\cos^2\alpha$.

substitute $u = \ln y$, $du = \frac{dy}{y}$ $\therefore \int \frac{dy}{y \ln y} = \int \frac{du}{u} = \ln|u| + c$
$= \ln|\ln y| + c$. So: $\ln|\ln y| + c = e^x$. Initial values: $\ln|\ln e^e| + c = e^0 = 1$
$= \ln e + c = 1 + c$, so $c = 0$, and
$\ln|\ln y| = e^x$, $|\ln y| = e^{e^x}$, $\ln y = +e^{e^x}$ (since $\ln y(0) > 0$), and $y = e^{e^{e^x}}$.

1-13

Solve dy/dx = x/sin y subject to the condition that y(2) = π.

****

Separating variables, $\sin y \, dy = x \, dx$.

Integrating, $-\cos y = \frac{1}{2}x^2 + C$.

Substituting $x = 2$ and $y = \pi$ gives $1 = 2 + C$
so $C = -1$. Using this value of C, you get
$\cos y = 1 - \frac{1}{2}x^2$, or even better,
$y = \cos^{-1}(1 - \frac{1}{2}x^2)$ for the solution.

Recall that the inverse cosine function has domain $[-1, 1]$, so we require that $-1 \leq (1 - \frac{1}{2}x^2) \leq 1$. Solving this inequality for x, we find that $-2 \leq x \leq 2$. Hence the solution has domain $[-2, 2]$.

1-15

Find the explicit solution $y = f(x)$ to the initial-value problem

$$(1 + x^2)(y + 1)^{1/2}dx + xdy = 0, \quad f(1)=3.$$

****

ASSUMING $y > -1$ AND CONSIDERING ONLY $x > 0$, THIS EQUATION IS SEPARABLE WITH INTEGRATING FACTOR $\mu(x,y) = \dfrac{1}{x(y+1)^{1/2}}$.

$$\left(\frac{1+x^2}{x}\right)dx + \frac{1}{(y+1)^{1/2}}dy = 0$$

$$\left(\frac{1}{x} + x\right)dx + (y+1)^{-1/2}dy = 0$$

$$\int\left(\frac{1}{x} + x\right)dx + \int(y+1)^{-1/2}dy = \int 0$$

$$\ln|x| + \tfrac{1}{2}x^2 + 2(y+1)^{1/2} = C$$

APPLY THE INITIAL CONDITIONS $\begin{cases} x = 1 \\ y = 3 \end{cases}$ TO FIND C:

$$\ln|1| + \tfrac{1}{2}(1)^2 + 2(3+1)^{1/2} = C$$

$$0 + \tfrac{1}{2} + 2\sqrt{4} = \tfrac{1}{2} + 4 = \tfrac{9}{2} = C$$

$$\Longrightarrow \quad \ln|x| + \tfrac{1}{2}x^2 + 2(y+1)^{1/2} = \frac{9}{2}$$

TO FIND THE EXPLICIT SOLUTION, WE SOLVE FOR y AS A FUNCTION OF x.

$$2\ln|x| + x^2 + 4\sqrt{y+1} = 9$$

$$4\sqrt{y+1} = 9 - x^2 - 2\ln|x|$$

$$16(y+1) = \left[9 - x^2 - 2\ln|x|\right]^2$$

$$y = \frac{1}{16}\left[9 - x^2 - 2\ln|x|\right]^2 - 1$$

To write dx in terms of α, we differentiate both sides of $\tan\alpha = x$ with respect to α, obtaining

$$\frac{1}{\cos^2\alpha} = \frac{dx}{d\alpha} \quad \text{so} \quad dx = \frac{1}{\cos^2\alpha}\,d\alpha .$$

Now the integral becomes

$$\int \cos^2\alpha \,\frac{1}{\cos^2\alpha}\,d\alpha = \int d\alpha = \alpha + C_1$$

(C_1: another constant.)

In conclusion, we have

$$y^2 + \underbrace{C_0 - C_1}_{\text{yet another constant; call it } C.} = \alpha$$

$$y^2 + C = \alpha$$

$$\overset{\tan\alpha = x}{=} \arctan x$$

Finally, taking tan of both sides

$$\tan(y^2 + C) = x .$$

This is the answer (D). Note that (D) is definitely different from (A) or (C). (B) is plainly wrong, and (E) is close but with a wrong sign. (E) will imply $\tan(-y^2 + C) = x$ which is not the answer.

1-17

Solve the differential equation $\frac{dy}{dx} = \frac{y^2 + 1}{x^2 - 1}$.

* *

Separating the variables x and y and integrating, we have

$$\int \frac{dy}{y^2+1} = \int \frac{dx}{x^2-1}. \qquad (*)$$

Note that the integral on the left hand side is $\tan^{-1} y$. The integral on the right hand side can be evaluated, using the method of partial fractions, as follows.

Let $$\frac{1}{x^2-1} = \frac{1}{(x+1)(x-1)} = \frac{A}{x+1} + \frac{B}{x-1}.$$

Multiplying through by $(x+1)(x-1)$ gives

$$A(x-1) + B(x+1) = 1.$$

Now using $x = 1$, we find $B = 1/2$

and with $x = -1$, we have $A = -1/2$.

Therefore $$\int \frac{dx}{x^2-1} = \frac{1}{2} \int \left\{ \frac{1}{x-1} - \frac{1}{x+1} \right\} dx$$

$$= \frac{1}{2} \left\{ \ln(x-1) - \ln(x+1) \right\} = \frac{1}{2} \ln \frac{x-1}{x+1}.$$

Thus (*) gives the solution

$$\tan^{-1} y = \frac{1}{2} \ln \frac{x-1}{x+1} + \text{const.}$$

1-16

Find the unique solution, if it exists, of the differential equation:

$(e^y + t^2e^y)\frac{dy}{dt} = t$, subject to the initial condition that when $t = 0$, $y = 0$.

****

Rewriting: $e^y(1+t^2)\frac{dy}{dt} = t$. Variables are separable.

(*) $e^y dy = \frac{t}{1+t^2}\,dt$; Integrating

$\Rightarrow \int e^y dy = \int \frac{t}{(1+t^2)}\,dt = \frac{1}{2}\int \frac{2t\,dt}{(1+t^2)} = \frac{1}{2}\int \frac{du}{u}$

$u = 1+t^2$

$du = 2t\,dt$

$\therefore \quad e^y = \frac{1}{2}\ln|u| + c$

$e^y = \frac{1}{2}\ln|1+t^2| + c$ and, since $1+t^2 > 0$

$\therefore \quad e^y = \frac{1}{2}\ln(1+t^2) + c$. Imposing the condition $t=0 \Rightarrow y=0$: $e^0 = \frac{1}{2}\ln(1) + c$

$1 = 0 + c \Rightarrow c = 1$

$\therefore e^y = \frac{1}{2}\ln(1+t^2) + 1$ or $y = \ln\left[\frac{1}{2}\ln(1+t^2) + 1\right]$.

Alternatively, integrating explicitly: (from (*))

$\int_0^y e^y dy = \frac{1}{2}\int_0^t \frac{2t}{(1+t^2)}\,dt \Rightarrow e^y\Big]_0^y = \frac{1}{2}\ln(1+t^2)\Big]_0^t$

$\therefore e^y - e^0 = \frac{1}{2}\ln(1+t^2) - \frac{1}{2}\ln 1$

hence, $e^y - 1 = \frac{1}{2}\ln(1+t^2) \Rightarrow e^y = \frac{1}{2}\ln(1+t^2) + 1$

$\therefore y = \ln\left[\frac{1}{2}\ln(1+t^2) + 1\right]$.

1-19

Solve the equation: $y'(x^2-2x+2) = xy-y$

****

Solution: $\frac{dy}{dx}(x^2-2x+2) = y(x-1)$

$\Rightarrow dy(x^2-2x+1+1) = y(x-1)\,dx$

$\Rightarrow \frac{dy}{y} = \frac{(x-1)\,dx}{(x-1)^2+1} \Rightarrow \frac{dy}{y} = \frac{(x-1)\,d(x-1)}{(x-1)^2+1}$

integrating: $\ln|y| = \frac{1}{2}\ln[(x-1)^2+1] + C_1$

exponentiating: $|y| = [(x-1)^2+1]^{1/2} e^{C_1}$

suggests: $y = C\sqrt{(x-1)^2+1}$. (1)

Verification: rewriting (1): $y[(x-1)^2+1]^{-1/2} = C$

differentiating: $y'[(x-1)^2+1]^{-1/2} - \frac{1}{2}y[(x-1)^2+1]^{-3/2}2(x-1) = 0$

multiplying by $[(x-1)^2+1]^{3/2}$: $y'[(x-1)^2+1] - y(x-1) = 0$

$\Rightarrow y'[x^2-2x+1+1] = y(x-1)$

$\Rightarrow y'(x^2-2x+2) = yx-y$

1-20

Solve $\frac{dN}{dt} = rN(1 - \frac{N}{K})$, $N(0) = N_0$, r, $K > 0$, $K \neq N_0$; and find the limit of $N(t)$ as $t \to +\infty$.

****

1-18

Determine the solution of the separable DE

$$\sin\theta\, dr - 2r\cos\theta\, d\theta = 0$$

such that $r = 1$ when $\theta = \pi/2$.

****

Separate variables:

$$\frac{dr}{r} - \frac{2\cos\theta}{\sin\theta}\, d\theta = 0.$$

Integrate:

$$\int \frac{dr}{r} - \int \frac{2\cos\theta}{\sin\theta}\, d\theta = 0.$$

$$\Rightarrow \int \frac{dr}{r} - \int 2\,\frac{d(\sin\theta)}{\sin\theta} = 0.$$

$$\ln|r| - 2\ln|\sin\theta| = C$$

$$\ln|r| = C + 2\ln|\sin\theta|$$

$$|r| = e^{C + 2\ln|\sin\theta|} = e^{C} e^{2\ln|\sin\theta|}$$

Let $e^{C} = |A|$

$$|r| = |A|\sin^2\theta$$

Then removing absolute values

$$r = A\sin^2\theta$$

Applying the initial conditions, $r = 1$ when $\theta = \frac{\pi}{2}$,

$$1 = A\sin^2\left(\frac{\pi}{2}\right) = A(1)^2 = A$$

$$\boxed{r = \sin^2\theta}$$

cross-multiply: $\frac{dy}{dx} = \frac{4x^3y}{3y^2 - x^4}$

gives $4x^3y\,dx = (3y^2 - x^4)\,dy$, or

$$\underbrace{4x^3y}_{M}\,dx + \underbrace{(x^4 - 3y^2)}_{N}\,dy = 0$$

Since $\frac{\partial M}{\partial y} = 4x^3 = \frac{\partial N}{\partial x}$, this is exact.

$$\psi(x,y) = \int^x 4x^3y\,dx = x^4y + h(y)$$

$\frac{\partial \psi}{\partial y} = x^4 + h'(y)$ must equal $N = x^4 - 3y^2$,

so $h'(y) = -3y^2$, and $h(y) = -y^3$ works.

$\therefore\ x^4y - y^3 = c$.

Initial values: $1^4(-1) - (-1)^3 = c = 0$, and

$x^4y = y^3$, so dividing by y $(y(1) \neq 0)$

gives $x^4 = y^2$, and $y = -x^2$ (since $y(1) < 0$).

1-22

Determine g(y) such that $g(y)e^x\,dx + (e^x + y)dy = 0$ will have a solution.

****

$M\,dx + N\,dy = 0$ will be exact if $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$. In this case $M = g(y)e^x$, $N = e^x + y$, for exactness $g'(y)e^x = e^x$. Thus $g'(y) = 1$ or $g(y) = y + C$ where C is a constant. $(y + C)e^x dx + (e^x + y)dy = 0$ will have a solution since it is exact.

$$\frac{K\,dN}{N(K-N)} = r\,dt \quad , \quad \left(\frac{1}{N} - \frac{1}{N-K}\right) dN = r\,dt$$

Integrating from 0 to t we get

$$\ln\left|\frac{N(t)}{N(t)-K}\right| - \ln\left|\frac{N_0}{N_0-K}\right| = rt$$

$$\frac{N_0-K}{N_0} \cdot \frac{N(t)}{N(t)-K} = e^{rt}$$

$$\left(1-\frac{K}{N_0}\right) e^{-rt} N(t) = N(t) - K$$

$$\left(1 - \left(1-\frac{K}{N_0}\right) e^{-rt}\right) N(t) = K$$

$$N(t) = \frac{K}{1 - \left(1-\frac{K}{N_0}\right) e^{-rt}}$$

Therefore $\lim_{t \to +\infty} N(t) = K$.

EXACT

1-21

Solve the initial value problem

$$y' = \frac{4x^3y}{3y^2 - x^4} \quad ; \quad y(1) = -1$$

****

With y^2 in the denominator it isn't linear, so

1-24

Solve the following differential equation :

$$(e^x y + \sin x)dx + (e^x + \cos y)dy = 0. \qquad (1)$$

* *

This equation is of the form

$$M(x,y)\,dx + N(x,y)\,dy = 0$$

where

$$M(x,y) = e^x y + \sin x, \quad N(x,y) = e^x + \cos y.$$

Since $\frac{\partial M}{\partial y} = e^x$ and $\frac{\partial N}{\partial x} = e^x$,

it follows that equation (1) is exact.

Accordingly, we can find a function $f(x,y)$ such that

$$\frac{\partial f}{\partial x} = M(x,y) = e^x y + \sin x \qquad (2)$$

and

$$\frac{\partial f}{\partial y} = N(x,y) = e^x + \cos y. \qquad (3)$$

Integrating equation (2) with respect to x, holding y fixed, we obtain

$$f(x,y) = e^x y - \cos x + c(y) \qquad (4)$$

where c is a function of y.

Differentiating equation (4) partially with respect to y, we find

$$\frac{\partial f}{\partial y} = e^x + \frac{dc}{dy}.$$

1-23

Find the value of the constant m so that the equation
$$(2xy^3 + y^m)\,dx + (3x^2y^2 + x)\,dy = 0$$
is exact. Solve the resulting exact equation.

****

The given equation is of the type
$M(x,y)\,dx + N(x,y)\,dy = 0$, which is exact if and only if $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$.

Here $M(x,y) = 2xy^3 + y^m$ and $N(x,y) = 3x^2y^2 + x$ are continuous functions with continuous partial derivatives of first order.

$$\frac{\partial M}{\partial y} = 6xy^2 + m y^{m-1}, \quad \frac{\partial N}{\partial x} = 6xy^2 + 1$$

The condition of exactness requires that
$$6xy^2 + m y^{m-1} = 6xy^2 + 1$$
giving $m = 1$.

The given diff. eq. becomes
$$(2xy^3 + y)\,dx + (3x^2y^2 + x)\,dy = 0$$
Now $M(x,y) = 2xy^3 + y$. We have to find a function $\varphi(x,y)$ such that $\frac{\partial \varphi}{\partial x} = M(x,y) = 2xy^3 + y$.

Integrating with respect to x, we obtain
$$\varphi(x,y) = \int (2xy^3 + y)\,dx$$
$$= x^2y^3 + xy + g(y)$$

We find $g(y)$ by using the relation
$\frac{\partial \varphi}{\partial y} = N(x,y)$
that is $3x^2y^2 + x + g'(y) = 3x^2y^2 + x$
$g'(y) = 0$ giving $g(y) = c_1$, constant
that is $\varphi(x,y) = x^2y^3 + xy + c_1$
Therefore the solution of the given differential equation is $x^2y^3 + xy + c_1 = c$

$$x^2y^3 + xy = C \quad (\text{where } C = c - c_1, \text{ is an arbitrary constant})$$

$\frac{\partial \psi}{\partial y} = x^2 + h'(y)$ must equal $N = x^2 - y$, so $h'(y) = -y$, and $h(y) = -y^2/2$ works.

$\therefore\ x^2 y - y^2/2 = c.$

Initial values: $1^2 \cdot 2 - 2^2/2 = c = 0$, and $x^2 y = y^2/2$. Dividing by y $(y(1) \neq 0)$ gives $x^2 = y/2$, or $y = 2x^2$.

1-26

Solve the differential equation:

$$(x + \sin y)\,dx + (x \cos y - 2y)\,dy = 0$$

****

The equation is <u>exact</u> since

$$\frac{\partial}{\partial y}(x + \sin y) = \cos y = \frac{\partial}{\partial x}(x \cos y - 2y)$$

Now, $F(x,y) = \int (x + \sin y)\,dx + g(y)$

$$= \frac{x^2}{2} + x \sin y + g(y)$$

Thus, $\frac{\partial F}{\partial y} = x \cos y + g'(y) = x \cos y - 2y$

So $g'(y) = -2y$ and hence $g(y) = -y^2 + C$

The general solution is:

$$\frac{x^2}{2} + x \sin y - y^2 = C$$

Using equation (3), we then have

$$e^x + \frac{dc}{dy} = e^x + \cos y$$

so that

$$\frac{dc}{dy} = \cos y .$$

This equation is easily integrated to give

$$c(y) = \sin y .$$

Substituting this into equation (4), we obtain the solution

$$e^x y - \cos x + \sin y = \text{const.}$$

1-25

Solve the initial value problem

$$y' = \frac{2xy}{y - x^2} , \quad y(1) = 2.$$

****

With y in the denominator it isn't linear, so

$$\frac{dy}{dx} = \frac{2xy}{y-x^2} \text{ gives } 2xy\,dx = (y-x^2)dy$$

after cross-multiplication, or

$$\underbrace{2xy}_{M}\,dx + \underbrace{(x^2-y)}_{N}\,dy = 0$$

Since $\frac{\partial M}{\partial y} = 2x = \frac{\partial N}{\partial x}$, this is exact.

$$\psi(x,y) = \int^x 2xy\,dx = x^2y + h(y)$$

1-28

Determine whether the differential equation is exact. If it is exact,

(a) find the solution, and

(b) indicate the region D of the xy plane in which we are guaranteed the existence and uniqueness of a solution

$(xy^2 + 3x^2y)dx + (x^3 + yx^2)dy = 0 \qquad y(1) = 1$

****

A differential equation, $M\,dx + N\,dy = 0$ is exact if $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$.

$\frac{\partial M}{\partial y} = 2xy + 3x^2$, $\frac{\partial N}{\partial x} = 3x^2 + 2xy$, hence it is exact.

(a) If exact, then there is a function, $f(x,y)$, such that

$df = M\,dx + N\,dy$; So, $\therefore \frac{\partial f}{\partial x} \overset{(*_1)}{=} xy^2 + 3x^2y$ & $\frac{\partial f}{\partial y} \overset{(*_2)}{=} x^3 + yx^2$

$\frac{\partial f}{\partial x} = M$ $\quad \frac{\partial f}{\partial y} = N$

$(*_1) \Rightarrow f = \frac{x^2y^2}{2} + x^3y + C(y)$ $(*_3)$

But $x^3 + yx^2 \overset{(*_2)}{=} \frac{\partial f}{\partial y} \overset{(*_3)}{=} x^2y + x^3 + C'(y)$

$\Rightarrow 0 = C'(y)$, $\therefore C(y) = K_1$

Since $df = 0 \Rightarrow f = K_2$ $(*_4)$

$\therefore f \overset{(*_3)}{=} \frac{x^2y^2}{2} + x^3y + K_1 \overset{(*_4)}{=} K_2$ (let $K = K_2 - K_1$)

$\therefore \frac{x^2y^2}{2} + x^3y = K$; If $y(1) = 1 \Rightarrow \frac{1}{2} + 1 = K$

or $K = \frac{3}{2}$; Hence, $\underline{\frac{x^2y^2}{2} + x^3y = \frac{3}{2}}$

(b)

$\frac{dy}{dx} = g(x,y) = -\frac{xy^2 + 3x^2y}{x^3 + yx^2}$; We have a unique solution in D, where $D = \begin{cases} g \text{ is continuous} \\ \frac{\partial g}{\partial y} \text{ is continuous} \end{cases}$

$g = -\frac{(xy^2 + 3x^2y)}{x^2(x+y)}$ is cont. as long as the denominator $\neq 0$.
i.e., as long as $x^2(x+y) \neq 0$

1-27

Find the value of b for which the following differential equation is EXACT,and then solve it, using that value of b.

$$(x\,y^2 + b\,x^2\,y)\,dx + (x + y)\,x^2\,dy = 0.$$

****

If we write the equation as

$$dF(x,y) = M(x,y)\,dx + N(x,y)\,dy = 0$$

then it is an exact differential iff

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.$$

Now $\frac{\partial M}{\partial y} = 2xy + bx^2$ and $\frac{\partial N}{\partial x} = 2xy + 3x^2$

and they are equal when $b = 3$. Equating

$$\frac{\partial F}{\partial x} = xy^2 + 3x^2y \quad \text{and} \quad \frac{\partial F}{\partial y} = x^3 + x^2y.$$

Integrating the above two expressions with respect to x and y respectively

$$F(x,y) = \frac{x^2y^2}{2} + x^3y + f(y)\,; \quad F(x,y) = \frac{x^3y^2}{2} + x^3y + g(x).$$

Comparing the two expressions for $F(x,y)$, we must have

$$f(y) = g(x) = C.$$

Therefore the solution is given by

$$\frac{x^2y^2}{2} + x^3y = C$$

for some arbitrary constant.

Therefore, we must have

$$18xy^2 + \cos y = 2kxy^2 + \cos y$$

$$18xy^2 = 2kxy^2$$

$$18 = 2k$$

$$9 = k$$

1-30

Solve the differential equation

$$\frac{dy}{dx} + \frac{e^x \sin y - 2y\sin x + 5x}{e^x \cos y + 2\cos x - \sec^2 y} = 0$$

****

$$(e^x \sin y - 2y\sin x + 5x)\,dx + (e^x \cos y + 2\cos x - \sec^2 y)\,dy = 0$$

$M_y = e^x \cos y - 2\sin x$

$N_x = e^x \cos y - 2\sin x$ so exact

seek $f(x,y)$ so that $\frac{\partial f}{\partial x} = M = e^x \sin y - 2y\sin x + 5x$

and $\frac{\partial f}{\partial y} = N = e^x \cos y + 2\cos x - \sec^2 y$

$$f(x,y) = \int (e^x \sin y - 2y\sin x + 5x)\,dx = e^x \sin y + 2y\cos x + \tfrac{5}{2}x^2 + g(y)$$

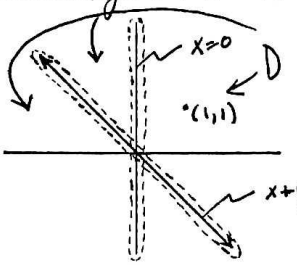
$$\frac{\partial f}{\partial y} = e^x \cos y + 2\cos x + g'(y)$$ so $g'(y) = -\sec^2 y$

$g(y) = -\tan y + C_1$

de solution: $e^x \sin y + 2y\cos x + \frac{5}{2}x^2 - \tan y = C$

$\frac{\partial g}{\partial y} = \frac{(\ )}{[x^2(x+y)]^2}$. Hence, the same condition as g for continuity. $\therefore D = \mathbb{R}^2 - \{x=0\} - \{x+y=0\}$

Graphically:



Note: $(1,1) \in D$

Hence, the solution in (a) is unique!!

1-29

Determine the constant k such that the given differential equation becomes exact.

$$(6xy^3 + \sin y)dx + (kx^2y^2 + x\cos y)dy = 0.$$

****

The equation is written in the standard form $M(x,y)dx + N(x,y)dy = 0$

we observe that

$$M(x,y) = 6xy^3 + \sin y \quad , \quad N(x,y) = kx^2y^2 + x\cos y$$

$$\frac{\partial M}{\partial y} = 18xy^2 + \cos y \quad , \quad \frac{\partial N}{\partial x} = 2kxy^2 + \cos y$$

A necessary and sufficient condition for exactness is $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$.

1-32

$(3xy^2 - 2y\cos x)dy + (y^3 + y^2\sin x - x)dx = 0.$

(A) $xy^3 + y^2\sin x - \frac{1}{2}x^2 = C$

(B) $xy^3 - y^2\sin x - \frac{1}{2}x^2 = C$

(C) $xy^3 - y^2\cos x - \frac{1}{2}x^2 = C$

(D) $xy^3 + y^2\cos x - \frac{1}{2}x^2 = C$

(E) $xy^3 + y^2\cos x + \frac{1}{2}x^2 = C.$

****

Denote the coefficients of dx and dy by $M(x,y)$ and $N(x,y)$ respectively: The equation is

$$\underbrace{(3xy^2 - 2y\cos x)}_{N(x,y)}dy + \underbrace{(y^3 + y^2\sin x - x)}_{M(x,y)}dx = 0$$

Check

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}(y^3 + y^2\sin x - x) = 3y^2 + 2y\sin x$$

$$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}(3xy^2 - 2y\cos x) = 3y^2 + 2y\sin x$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, the equation is exact. This means that there is a function $u(x,y)$ ("potential") so that

$$\frac{\partial u}{\partial x} = M = y^3 + y^2\sin x - x \quad \left(\text{and also, } \frac{\partial u}{\partial y} = N\right).$$

(Furthermore, it is known that the ("equipotential") curves $u(x,y) = \text{Const}$ are the solutions of the exact equation)

Integrating $\frac{\partial u}{\partial x}$ with respect to x we obtain

1-31

Show that the DE $(2xy - y^2)dx - (2xy - x^2)dy = 0$ is exact and solve it.

****

The exact equation may be written as

$M\,dx + N\,dy = 0$, where $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$,

Here $M = 2xy - y^2$, $N = -2xy + x^2$

$\frac{\partial M}{\partial y} = 2x - 2y$, $\frac{\partial N}{\partial x} = -2y + 2x$

Method 1 Solution by integration

$$F(x,y) = \int M\,dx + \varphi(y)$$
$$= \int (2xy - y^2)dx + \varphi(y)$$
$$= x^2y - xy^2 + \varphi(y)$$

$$\frac{\partial F}{\partial y} = x^2 - 2xy + \varphi'(y) = N = -2xy + x^2$$

Hence $\varphi'(y) = 0 \Rightarrow \varphi = C_1$

Solution $\boxed{F(x,y) = x^2y - xy^2 = C}$

Method 2 Solution by grouping

$$(2xy - y^2)dx - (2xy - x^2)dy = 0$$

$$2xy\,dx - y^2dx - 2xy\,dy + x^2dy = 0.$$

The indicated terms group together.

$d(x^2y) - d(y^2x) = 0 \Rightarrow \boxed{x^2y - xy^2 = C}$

1-33

Solve the equation: $\dfrac{4xy\sin(x)\cos(y)+2x^2y\cos(x)\cos(y)}{2x^2\sin(x)\cos(y)-2x^2y\sin(x)\sin(y)} + \dfrac{dy}{dx} = 0$

****

Solution: $\underbrace{(4xy\sin x\cos y + 2x^2y\cos x\cos y)}_{M}dx + \underbrace{(2x^2\sin x\cos y - 2x^2y\sin x\sin y)}_{N}dy = 0$

Then $\dfrac{\partial M}{\partial y} = 4x\sin x\cos y - 4xy\sin x\sin y + 2x^2\cos x\cos y - 2x^2y\cos x\sin y$

and $\dfrac{\partial N}{\partial x} = 4x\sin x\cos y + 2x^2\cos x\cos y - 4xy\sin x\sin y - 2x^2y\cos x\sin y$, so it's exact.

$\Rightarrow F = \int M\,dx = \int (4xy\sin x\cos y + 2x^2y\cos x\cos y)dx + T(y)$

$\Rightarrow F = 2y(\cos y)x^2\sin x + T(y)$. Then $\dfrac{\partial F}{\partial y} = N$

$\Rightarrow 2\cos y\,x^2\sin x - 2yx^2\sin x\sin y + T'(y)$

$= 2x^2\sin x\cos y - 2x^2y\sin x\sin y \Rightarrow T'(y) = 0$

$\Rightarrow F = 2x^2y\sin x\cos y = c$ is the implicit solution. ∎

$$u(x,y) = xy^3 - y^2\cos x - \tfrac{1}{2}x^2 + g(y)$$

($g(y)$: some function of y)

Then,

$$\frac{\partial u}{\partial y} = 3xy^2 - 2y\cos x + g'(y).$$

(on the other hand $\frac{\partial u}{\partial y} = N$)

$$N = 3xy^2 - 2y\cos x$$

Hence,

$$g'(y) = 0$$

$$g(y) = C_0 \quad \text{(some constant)}$$

$$u(x,y) = xy^3 - y^2\cos x - \tfrac{1}{2}x^2 + C_0$$

The solutions to the exact equation are the curves

$$u(x,y) = C_1 \quad \text{(some constant)}$$

Substituting $u(x,y)$, we obtain for the solutions

$$xy^3 - y^2\cos x - \tfrac{1}{2}x^2 + C_0 = C_1.$$

Hence,

$$xy^3 - y^2\cos x - \tfrac{1}{2}x^2 = \underbrace{C_1 - C_0}_{\text{another constant, call it } C}$$

The correct answer is (C).

1-35

Check $6x^2y^2dx + (4x^3y + 2y-1)dy = 0$ for exactness and then solve.

****

$M(x,y)\,dx + N(x,y)\,dy = 0$ is exact if $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$.

$M = 6x^2y^2 \qquad \frac{\partial M}{\partial y} = 12x^2y$

$N = 4x^3 + 2y - 1 \qquad \frac{\partial N}{\partial x} = 12x^2y$

The DE is exact. There is a function $\phi(x,y)$ so that $\frac{\partial \phi}{\partial x} = M$ and $\frac{\partial \phi}{\partial y} = N$.

$\frac{\partial \phi}{\partial x} = 6x^2y^2$ gives $\phi(x,y) = 2x^3y^2 + h(y)$

To find $h(y)$ we use $\frac{\partial \phi}{\partial y} = N$.

$\frac{\partial \phi}{\partial y} = 4x^3y + h'(y) = N = 4x^3y + 2y - 1$

So $h'(y) = 2y - 1$ and $h(y) = y^2 - y$.

$\phi(x,y) = 2x^3y^2 + y^2 - y$. The DE is equivalent to $d\phi = 0$ so the general solution is $\phi = C$ (constant). Our solution is then

$$2x^3y^2 + y^2 - y = C.$$

1-34

Solve the initial value problem $dy/dx = (1-y\sin xy)/(x\sin xy+y^2)$, $y(0)=1$.

****

This had better be exact! Rewrite the equation as

$(x\sin xy + y^2)\,dy = (1-y\sin xy)\,dx$, or

$(x\sin xy + y^2)\,dy + (y\sin xy - 1)\,dx = 0$.

If a potential function ϕ exists, it satisfies

$\frac{\partial\phi}{\partial y} = x\sin xy + y^2$ and $\frac{\partial\phi}{\partial x} = y\sin xy - 1$.

Integrating $\frac{\partial\phi}{\partial y}$ with respect to y,

$$\phi = \int x\sin xy + y^2\,dy = x\left(-\frac{\cos xy}{x}\right) + \frac{y^3}{3} + C(x)$$

$$\phi = -\cos xy + y^3/3 + C(x)$$

Integrating $\frac{\partial\phi}{\partial x}$ with respect to x,

$$\phi = \int y\sin xy - 1\,dx = y\left(-\frac{\cos xy}{y}\right) - x + C(y)$$

$$\phi = -\cos xy - x + C(y)$$

The terms containing x and y (i.e., $-\cos xy$) match, so the equation is exact, and by inspection $C(x) = -x$, $C(y) = \frac{y^3}{3}$. Then $\phi = -\cos xy + \frac{y^3}{3} - x$, and the general solution is $\phi = C$, or $-\cos xy + \frac{y^3}{3} - x = C$. With $x=0$ and $y=1$, $-1 + \frac{1}{3} = C$, or $C = \frac{2}{3}$. The solution is

$-\cos xy + \frac{y^3}{3} - x = \frac{2}{3}$.

1-37

Show that the following differential equation

$$(y \cos x + 2x e^y)\,dx + (\sin x + x^2 e^y + 2)\,dy = 0$$

is an exact differential. Hence find the general solution.

****

Write the equation in the form

$$M(x,y)\,dx + N(x,y)\,dy = 0$$

where

$$M(x,y) = y\cos x + 2xe^y$$

$$N(x,y) = \sin x + x^2 e^y + 2.$$

The differential equation is said to be an exact differential if there exists a function $F(x,y)$ such that

$$dF = \frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial y}dy = M(x,y)dx + N(x,y)dy$$

then the solution is given by $F(x,y) = C$, C is an arbitrary constant. Since

$$\frac{\partial}{\partial y}\left(\frac{\partial F}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial y}\right)$$

this requires

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

for exact differential. Now

$$\frac{\partial M}{\partial y} = \cos x + 2xe^y$$

and

$$\frac{\partial N}{\partial x} = \cos x + 2xe^y = \frac{\partial M}{\partial y}$$

so the given equation is an exact differential.

1-36

Solve the initial-value problem:

$$(x \cos 4y + y^2 - 6x^2)dx + (2xy - 2x^2 \sin 4y)dy = 0, \quad y(1)=0$$

****

$$M = x\cos 4y + y^2 - 6x^2 \qquad N = 2xy - 2x^2 \sin 4y$$

$$\frac{\partial M}{\partial y} = -4x\sin 4y + 2y \qquad \frac{\partial N}{\partial x} = 2y - 4x\sin 4y$$

$\therefore$ THIS EQUATION IS EXACT.

$$\Longrightarrow \underbrace{x\cos 4y\,dx}_{(A)} + \overbrace{y^2dx - 6x^2dx}^{(B)} + \overbrace{2xy\,dy}^{(B)} \underbrace{- 2x^2\sin 4y\,dy}_{(A)} = 0$$

$$\underset{(A)}{d\left(\tfrac{1}{2}x^2\cos 4y\right)} + \underset{(B)}{d(xy^2)} - d(2x^3) = d(C)$$

INTEGRATING TERM BY TERM, WE OBTAIN THE GENERAL IMPLICIT SOLUTION:

$$\tfrac{1}{2}x^2\cos 4y + xy^2 - 2x^3 = C$$

APPLY INITIAL CONDITIONS $\begin{cases} x=1 \\ y=0 \end{cases}$ TO FIND C:

$$\tfrac{1}{2}(1)^2\cos 0 + 1(0)^2 - 2(1)^3 = C$$

$$\tfrac{1}{2} + 0 - 2 = -\tfrac{3}{2} = C$$

$\therefore$ THE SPECIFIC SOLUTION IS GIVEN BY

$$\tfrac{1}{2}x^2\cos 4y + xy^2 - 2x^3 = -\tfrac{3}{2},$$

OR $$x^2\cos 4y + 2xy^2 - 4x^3 + 3 = 0.$$

1-38

Solve (implicitly) the differential equation

$$3x^2(1 + \ln y)\,dx = \left(2y - \frac{x^3}{y}\right)dy.$$

****

In standard form we have

$$3x^2(1+\ln y)\,dx + \left(\frac{x^3}{y} - 2y\right)dy = 0.$$

Let $M(x,y) = 3x^2(1+\ln y)$ and $N(x,y) = x^3/y - 2y$.

$M_y(x,y) = 3x^2/y$ and $N_x(x,y) = 3x^2/y$.

Since $M_y = N_x$, the D.E. is exact and there exists a function $\varphi = \varphi(x,y)$ for which $d\varphi = M\,dx + N\,dy$.

If $\varphi_x = M$, then $\varphi = x^3(1+\ln y) + h(y)$ and

$\varphi_y = x^3/y + h'(y) = N = x^3/y - 2y$.

$h'(y) = -2y$; or, $h(y) = -y^2$.

$\varphi(x,y) = x^3(1+\ln y) - y^2$.

The (implicit form) solution of the D.E. is $x^3(1+\ln y) - y^2 = C$.

Now

$$\frac{\partial F}{\partial x} = M = y\cos x + 2xe^y$$

and on integration with respect to x

$$F(x,y) = y\sin x + x^2e^y + g(y)$$

for some arbitrary function $g(y)$. Differentiate the above expression with respect to y

$$\frac{\partial F}{\partial y} = \sin x + x^2e^y + \frac{dg}{dy}$$

and $\partial F/\partial y$ is equated to $N(x,y) = \sin x + x^2e^y + 2$

This forces $\frac{dg}{dy} = 2$ and so $g(y) = 2y$. Hence the solution to the given exact differential equation

$$y\sin x + x^2e^y + 2y = C.$$

we obtain

$$f(x,y) = \sin x \cos y + \ln x + g(y) \qquad (3)$$

where $g(y)$ is some unknown function of y. In order to find $g(y)$, we differentiate (3) with respect to y (regard x as a constant) to get

$$\frac{\partial f}{\partial y} = -\sin x \sin y + g'(y)$$

Compare this with (2), we find $g'(y) = \frac{1}{y}$. So

$$g(y) = \ln y \qquad (4)$$

Substitute (4) into (3)

$$f(x,y) = \sin x \cos y + \ln x + \ln y$$

Therefore the general solution of the equation is

$$\sin x \cos y + \ln x + \ln y = C$$

with any constant C.

1-40

Find a general solution: $(2xy-y-1)dx + (x^2-x)dy = 0$

****

This is of the form $M dx + N dy = 0$ where

$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} = 2x-1$. So this is exact.

1-39

Find the general solution of the equation

$$(\cos x\cos y + 1/x)dx + (1/y - \sin x\sin y)dy = 0$$

****

(Before solving an equation, it's important to know what type the equation is. The general order of type examination for a first order equation may be: Is the equation is separable ? homogeneous ? linear ? exact ? If our equation is none of the above, then try to use integrating factors or some differential formulas)

Now our equation is neither separable nor homogeneous. Check if it is exact.

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}\left(\cos x\cos y + \frac{1}{x} \right) = -\cos x\sin y$$

$$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}\left(\frac{1}{y} - \sin x\sin y \right) = -\cos x\sin y$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, the equation is exact. Hence the solution will be of the form

$$f(x, y) = C$$

with $$\frac{\partial f}{\partial x} = M = \cos x\cos y + \frac{1}{x} \qquad (1)$$

$$\frac{\partial f}{\partial y} = N = \frac{1}{y} - \sin x\sin y \qquad (2)$$

Integrate (1) with respect to x (regard y as a constant)

This equation is separable and gives

$$\frac{du}{\sqrt{1-u^2}} = \frac{dx}{x}$$

Thus $\sin^{-1}(u) = \ln x + c$

Substituting y/x for u we have

$$\sin^{-1}(y/x) = \ln x + c$$

Or $$\underline{\underline{y = x \sin(\ln x + c)}}$$

1-42

Solve the differential equation $\frac{dy}{dx} = \frac{x^2 + xy + y^2}{x(x+y)}$. (1)

* *

Let $p(x,y) = x^2 + xy + y^2$, $q(x,y) = x(x+y)$.

Note that $p(tx, ty) = t^2x^2 + t^2xy + t^2y^2$

$= t^2(x^2 + xy + y^2) = t^2 p(x,y)$

and

$q(tx, ty) = tx(tx + ty)$

$= t^2 x(x+y) = t^2 q(x,y)$.

In other words, the functions $p(x,y)$ and $q(x,y)$ are homogeneous of degree two. Therefore equation (1) is a homogeneous equation of degree two.

The solution is $f(x,y) = C$ where

$$\frac{\partial f}{\partial x} = 2xy - y + 1 \quad \text{and} \quad \frac{\partial f}{\partial y} = x^2 - x$$

$$\therefore f(x,y) = x^2y - xy + \phi(x)$$

$$\frac{\partial f}{\partial x} = 2xy - y + \phi'(x)$$

$$\therefore 2xy - y + \phi'(x) = 2xy - y + 1$$

$$\phi'(x) = 1$$

$$\phi(x) = x$$

The general solution is $x^2y - xy + x = C$

HOMOGENEOUS

1-41

Solve the homogeneous differential equation:

$$x\,dy - \left(y + \sqrt{x^2 - y^2}\right) dx = 0$$

****

Since the equation is homogeneous we let $y = ux$.

Then $dy = u\,dx + x\,du$, and substitution gives

$$x(u\,dx + x\,du) - (ux + \sqrt{x^2 - ux^2})\,dx = 0$$

1-43

Find the general solution of the equation

$$y^2dx - x(x+y)dy = 0$$

****

Since both y^2 (the coefficient of dx) and $-x(x+y)$ (the coefficient of dy) are homogeneous polynomials of degree 2, this is a homogeneous equation. Write the equation as

$$\frac{dy}{dx} = \frac{y^2}{x(x+y)} = \frac{(\frac{y}{x})^2}{1+\frac{y}{x}} \qquad (1)$$

Let $z = \frac{y}{x}$, then $y = xz$ and $\frac{dy}{dx} = z + x\frac{dz}{dx}$.

Now equation (1) becomes

$$z + x\frac{dz}{dx} = \frac{z^2}{1+z}$$

or

$$x\frac{dz}{dx} = -\frac{z}{1+z}$$

Separate the variables $\quad \frac{1+z}{z}dz = -\frac{dx}{x}$

Integrate both side $\quad \ln z + z = -\ln x + \ln c$

or $\quad \ln(ze^z) = \ln\frac{c}{x}$

that is $\quad ze^z = \frac{c}{x}$

Replace z by $\frac{y}{x}$, we obtain the solution

$$ye^{\frac{y}{x}} = c.$$

Remark: For such a homogeneous equation, if you make a change of variable $z = \frac{y}{x}$ (or $z = \frac{x}{y}$), you always obtain a separable equation with respect to x and z (or y and z).

We now set $u = \frac{y}{x}$, i.e. $y = xu$,

then $\frac{dy}{dx} = x\frac{du}{dx} + u$.

Substituting this in equation (1), we have

$$x\frac{du}{dx} + u = \frac{1+u+u^2}{1+u}$$

so that

$$x\frac{du}{dx} = \frac{1+u+u^2}{1+u} - u = \frac{1}{1+u}.$$

Separating the variables u and x, we obtain

$$(1+u)\,du = \frac{dx}{x}$$

which can be easily integrated to give

$$u + \frac{u^2}{2} = \ln x + c,$$

c being an arbitrary constant of integration.

Replacing u by y/x, we finally have

$$\frac{y}{x} + \frac{y^2}{2x^2} = \ln x + c.$$

This last equation can be further simplified to yield

$$y^2 + 2xy = 2x^2(\ln x + c).$$

1-45

Show that the following DE is homogeneous and solve it.

$$(2x + y)dx + (x - 2y)dy = 0.$$

****

$\frac{dy}{dx} = \frac{2x+y}{-x+2y} = \frac{2+(\frac{y}{x})}{-1+2(\frac{y}{x})} = F(\frac{y}{x})$ Homogeneous form

Hence set $y = vx$. Then $\frac{dy}{dx} = v + x\frac{dv}{dx}$

Substituting into the DE find

$$v + x\frac{dv}{dx} = \frac{2+v}{-1+v}$$

$$\Rightarrow \quad x\frac{dv}{dx} = \frac{2+v}{-1+2v} - v = \frac{2+v+v-2v^2}{-1+2v}$$

$$x\frac{dv}{dx} = \frac{2+2v+2v^2}{-1+2v}$$

The equation is now separable.

$$\frac{-1+2v}{2+2v-2v^2}\,dv = \frac{dx}{x}$$

Integrate both sides to find:

$$-\tfrac{1}{2}\ln|2+2v-2v^2| = \ln|x| + C$$

Replace v by $\frac{y}{x}$ and combine the logs to get

$$\tfrac{1}{2}\ln|2x^2+2xy-2y^2| + C = 0$$

$$\Rightarrow \quad \boxed{x^2+xy-y^2 = K,}$$

where $K = e^{-2C-\ln 2}$. This may also be solved as an exact equation.

1-44

Solve the following initial value problem

$$x\frac{dy}{dx} = y + \sqrt{x^2+y^2}, \quad y(1) = 0$$

by observing that the equation can be written as

$$\frac{dy}{dx} = F\left(\frac{y}{x}\right).$$

****

The equation can be written as

$$\frac{dy}{dx} = \frac{y}{x} + \sqrt{1+\left(\frac{y}{x}\right)^2}.$$

It is natural to put $V = y/x$ so that

$$\frac{dy}{dx} = x\frac{dv}{dx} + V = V + \sqrt{1+V^2}.$$

Cancelling V and separation of variable

$$\frac{dx}{x} = \frac{dV}{\sqrt{1+V^2}}$$

and upon integration

$$\ln x = \ln\left|V + \sqrt{1+V^2}\right| + C.$$

Using the initial condition : $x=1$, $V=0$; this gives $C=0$. Since $V+\sqrt{1+V^2}$ is always positive, we have

$$\ln x = \ln\left(V + \sqrt{1+V^2}\right)$$

or

$$x = V + \sqrt{1+V^2}$$

On simplification, we obtain

$$x^2 = y + \sqrt{x^2+y^2}$$

$$y = \frac{x^2-1}{2}.$$

1-47

Find the general solution: $(y^2 + 5xy)dx - x^2dy = 0.$

****

THE TOTAL (COMBINED) DEGREE OF EACH TERM IS 2, SO THIS IS A HOMOGENEOUS EQUATION. WE SET $y = xv$, SO THAT $dy = x\,dv + v\,dx$, AND SUBSTITUTE:

$$(x^2v^2 + 5x^2v)dx - x^2(x\,dv + v\,dx) = 0$$
$$x^2(v^2 + 5v)dx - x^3\,dv - x^2v\,dx = 0$$
$$x^2(v^2 + 4v)dx - x^3\,dv = 0$$

NOW SEPARATE VARIABLES AND INTEGRATE:

$$\frac{1}{x}dx - \frac{1}{v^2+4v}dv = 0$$
$$\int \frac{1}{x}dx - \int \frac{1}{v(v+4)}dv = \int 0$$

(PARTIAL FRACTIONS: $\frac{1}{v(v+4)} = \frac{A}{v} + \frac{B}{v+4} \Longrightarrow A = 1/4,\ B = -1/4$)

$$\int \frac{1}{x}dx - \int\left[\frac{1/4}{v} - \frac{1/4}{v+4}\right] = \int 0$$
$$\ln|x| - \tfrac{1}{4}\ln|v| + \tfrac{1}{4}\ln|v+4| = C_1$$
$$4\ln|x| - \ln|v| + \ln|v+4| = C_2$$
$$\ln\left|\frac{x^4(v+4)}{v}\right| = C_2$$
$$\frac{x^4(v+4)}{v} = e^{C_2} = C$$

SUBSTITUTING $y = xv$ (OR $v = y/x$) WE OBTAIN THE DESIRED SOLUTION:

$$\frac{x^4(y/x + 4)}{y/x} = \frac{x^4(y+4x)}{y} = C$$
$$\Longrightarrow x^4y + 4x^5 = Cy$$
$$\Longrightarrow y = \frac{4x^5}{C - x^4}$$

1-46

Solve $\frac{dy}{dt} = \frac{2t + 3y}{2t + y}$

****

Since $2t+3y$ and $2t+y$ are both homogeneous and of first degree, the given diff. eq. is homogeneous.

Substitute $y = vt$, where $v = \frac{y(t)}{t}$ is a function of t.

Therefore $$v + t\frac{dv}{dt} = \frac{2t+3vt}{2t+vt} = \frac{2+3v}{2+v}$$

$$t\frac{dv}{dt} = \frac{2+3v}{2+v} - v = \frac{2+v-v^2}{2+v}$$

Then

$$\frac{(2+v)}{v^2-v-2}\,dv + \frac{dt}{t} = 0$$

$$\int \frac{(2+v)\,dv}{(v-2)(v+1)} + \int \frac{dt}{t} = C_1, \text{ a constant}$$

$$\int \frac{4/3\,dv}{v-2} + \int \frac{-1/3\,dv}{v+1} + \int \frac{1}{t}\,dt = C_1$$

Integrating

$$\frac{4}{3}\ln|v-2| - \frac{1}{3}\ln|v+1| + \ln|t| = C_1$$

$$4\ln\left|\frac{y}{t}-2\right| - \ln\left|\frac{y}{t}+1\right| + \ln|t|^3 = C_2 \qquad C_2 = 3C_1$$

that is $$\ln\left|\left(\frac{y-2t}{t}\right)^4 t^3 \Big/ \frac{y+t}{t}\right| = C_2$$

Therefore $$(y-2t)^4 = C(y+t) \qquad C = e^{C_2} > 0$$

is the solution.

1-49

Solve by two methods: $(3y - 2x)y' + 4x - 2y = 0$.

****

A. Since $4\lambda x - 2\lambda y = \lambda(4x-2y)$ and $3\lambda y - 2\lambda x = \lambda(3y-2x)$, let us rewrite the given equation as $(4x-2y)dx + (3y-2x)dy = 0$ and note that the coefficients of dx and dy are both homogeneous of degree 1, so either $y = vx$ or $x = vy$ will, by substitution, lead to a "variables separable" equation. Choosing $y = vx$, $dy = v\,dx + x\,dv$ and the rewritten equation becomes

$$(4x-2vx)dx + (3vx-2x)(v\,dx + x\,dv) = 0.$$

$$x(4-2v)dx + x[(3v-2)v\,dx + (3v-2)x\,dv] = 0$$

$$[(4-2v) + v(3v-2)]dx + (3v-2)x\,dv = 0$$

$$\frac{dx}{x} + \frac{3v-2}{3v^2-4v+4}dv = 0. \quad \text{Integrating both sides,}$$

$$\ln|x| + \tfrac{1}{2}\ln|3v^2-4v+4| = K_1, \quad \text{or} \quad |x|\sqrt{3v^2-4v+4} = K_2$$

or $x^2(3v^2-4v+4) = K_3$. Note that $K_2 = e^{K_1}$, $K_3 = K_2^2$.

Since $y = vx$, $x^2\left(3\frac{y^2}{x^2} - 4\frac{y}{x} + 4\right) = 3y^2 - 4xy + 4x^2 = K_3$.

B. If $(4x-2y)dx + (3y-2x)dy = 0$, $\frac{\partial}{\partial y}(4x-2y) = -2 = \frac{\partial}{\partial x}(3y-2x)$ tells us that the equation is "exact".

$\int(4x-2y)dx = 2x^2 - 2xy + \psi(y)$, $\psi(y)$ an arbitrary function of y.

$\frac{\partial}{\partial y}[2x^2 - 2xy + \psi(y)] = -2x + \psi'(y) = 3y - 2x$. Then $\psi'(y) = 3y$, and $\psi(y) = \frac{3y^2}{2}$. $\therefore$ $2x^2 - 2xy + \frac{3y^2}{2} = K_4$ is a solution.

Similarly, $\int(3y-2x)dy = \frac{3}{2}y^2 - 2xy + \phi(x)$, ϕ arbitrary.

$\frac{\partial}{\partial x}\left[\frac{3}{2}y^2 - 2xy + \phi(x)\right] = -2y + \phi'(x) = 4x - 2y$. Then $\phi'(x) = 4x$, and $\phi(x) = 2x^2$. $\therefore$ $\frac{3}{2}y^2 - 2xy + 2x^2 = K_4$ is again a solution.

1-48

Solve the equation: $(3xy-3x^2+y^2)dx - x^2dy = 0$

****

Solution: It's homogeneous and the coefficient of dy is simpler, so let $y=ux \Rightarrow dy = u\,dx + x\,du$ and $u = y/x \Rightarrow (3xux - 3x^2 + u^2x^2)dx - x^2(u\,dx + x\,du) = 0$

$\Rightarrow x^2(3u-3+u^2)\,dx - x^2(u\,dx + x\,du) = 0$

$\Rightarrow (u^2+3u-3-u)\,dx - x\,du = 0$

$\Rightarrow (u^2+2u-3)\,dx - x\,du = 0 \Rightarrow \frac{dx}{x} - \frac{du}{x^2+2u-3} = 0$

$\Rightarrow \frac{dx}{x} + \frac{-du}{(u+3)(u-1)} = 0$. Then $\frac{-1}{(u+3)(u-1)} = \frac{a}{u+3} + \frac{b}{u-1}$

$\Rightarrow -1 = a(u-1) + b(u+3)$, if $u=1 \Rightarrow -1 = 4b \Rightarrow b = -\frac{1}{4}$

if $u=-3 \Rightarrow -1 = -4a \Rightarrow a = \frac{1}{4}$

$\Rightarrow \frac{dx}{x} + \left(\frac{1/4}{u+3} - \frac{1/4}{u-1}\right)du = 0$, integrating:

$\Rightarrow \ln|x| + \frac{1}{4}\ln|u+3| - \frac{1}{4}\ln|u-1| = c_1$

$\Rightarrow \ln|x| + \ln|u+3|^{1/4} + \ln|u-1|^{-1/4} = c_1$

$\Rightarrow \ln\left|x(u+3)^{1/4}(u-1)^{-1/4}\right| = c_1$, exponentiating:

$\Rightarrow \left|x(u+3)^{1/4}(u-1)^{-1/4}\right| = e^{c_1} = c_2$, raising to 4th power:

$\Rightarrow x^4(u+3)(u-1)^{-1} = c_2^4 = c_3$, substituting $u = y/x$:

$\Rightarrow x^4\left(\frac{y}{x}+3\right)\left(\frac{y}{x}-1\right)^{-1} = c_3 \Rightarrow x^4\,\frac{y+3x}{x}\left(\frac{y-x}{x}\right)^{-1} = c_3$

$\Rightarrow x^4\,\frac{y+3x}{x}\,\frac{x}{y-x} = c_3 \Rightarrow x^4(y+3x) = c(y-x)$

The equation becomes

$$\frac{du}{dy}y + u = u + \frac{1}{u}$$

That is $$\frac{du}{dy}y = \frac{1}{u}.$$

Here, the variables can be easily separated:

$$u\,du = \frac{1}{y}\,dy$$

Integrating both sides.

$$\frac{1}{2}u^2 + \underbrace{C_1}_{\text{constant}} = \ln y + \underbrace{C_2}_{\text{constant}}.$$

$$\frac{1}{2}u^2 = \ln y + \underbrace{C_2 - C_1}_{\text{"}}$$

$$(C_2 - C_1)\ln e = \ln \underbrace{e^{C_2 - C_1}}_{\text{yet another constant, call it } C}$$

$$\frac{1}{2}u^2 = \ln y + \ln C$$

$$= \ln Cy.$$

$$u^2 = 2\ln Cy.$$

$$\left(\frac{x}{y}\right)^2 = 2\ln Cy.$$

The answer (B) gives the solution in an implicit form Finally, the reader who is not aware yet of how crucial signs are in differential equations is invited to try $\frac{dx}{dy} = \frac{y}{x} - \frac{x}{y}$.

1-50

$\frac{dx}{dy} = \frac{x}{y} + \frac{y}{x}$

(A) $(y/x)^2 = 2\ln Cy$

(B) $(x/y)^2 = 2\ln Cy$

(C) $(x/y)^2 = -2\ln Cy$

(D) $(y/x)^2 = -2\ln Cy$

(E) $(x/y)^2 = 2\ln Cx$

****

Variables cannot be easily separated here.

Also $\underbrace{1}_{M}\,dx - \underbrace{\left(\frac{y}{x} + \frac{x}{y}\right)}_{N} dy = 0$ is not exact $\left(\frac{\partial M}{\partial y} = 0 \neq \frac{\partial N}{\partial x}\right)$.

Rewriting it, $\frac{1}{y}\frac{dx}{dy} - \frac{1}{y^2}x = \frac{1}{x}$ is not linear (because of the $\frac{1}{x}$ term)

We denote $f(x,y) = \frac{y}{x} + \frac{x}{y}$.

Then, $f(tx, ty) = \frac{ty}{tx} + \frac{tx}{ty} = \frac{y}{x} + \frac{x}{y} = f(x,y)$

Thus, the equation is homogeneous and the substitution $u = \frac{x}{y}$ will lead to an equation that will be separable.

We note that this substitution is helpful and may be tried even if an equation is not homogeneous. The substitution means that there is a function $u(y)$ so that $x(y) = u(y)\cdot y$.

Now, the left hand side becomes

$$\frac{x}{y} + \frac{y}{x} = u + \frac{1}{u},$$

while by the product rule for differentiation applied to $u\cdot y$,

$$\frac{dx}{dy} = \frac{du}{dy}y + u.$$

a. Since $M_y \neq N_x$, the D.E. is not exact.

b. Test for an integrating factor of the form $\mu = \mu(x)$.

$$\frac{M_y - N_x}{N} = \frac{4xy - 9y^2 + 3y^2}{7 - 3xy^2}$$

$$= \frac{2y(2x - 3y^2)}{7 - 3xy^2} \neq f(x).$$

Test for an integrating factor of the form $\mu = \mu(y)$.

$$\frac{N_x - M_y}{M} = \frac{-3y^2 - 4xy + 9y^2}{2xy^2 - 3y^3}$$

$$= \frac{-2y(2x - 3y)}{y^2(2x - 3y)} = -\frac{2}{y} = g(y).$$

$$\mu(y) = \exp\left(\int^y -\frac{2}{t}\,dt\right) = \exp(-2\ln y) = y^{-2}$$

c. $\mu M + \mu N = 0$: $(2x - 3y)\,dx + (7/y^2 - 3x)\,dy = 0$ is an exact D.E. Find $\varphi = \varphi(x, y)$ such that $d\varphi = \mu M\,dx + \mu N\,dy$.

$$\varphi_x = \mu M = 2x - 3y;\ \varphi = x^2 - 3xy + h(y)$$

$$\varphi_y = -3x + h'(y) = \mu N = 7/y^2 - 3x.$$

$$h'(y) = 7/y^2;\ h(y) = -7/y.$$

$$\varphi(x, y) = x^2 - 3xy - 7/y.$$

Solution (implicit form) of the D.E. is

$$x^2 - 3xy - 7/y = C.$$

1-51

Find a general solution: $(x^2+y^2)dx = xydy$

****

Let $y = vx$ $\quad dy = v\,dx + x\,dv$

$$(x^2 + v^2x^2)\,dx = vx^2(v\,dx + x\,dv)$$

$$(x^2 + v^2x^2 - v^2x^2)\,dx = vx^3\,dv$$

$$x^2\,dx = vx^3\,dv$$

$$\frac{dx}{x} = v\,dv$$

$$\ln|x| - \ln|c| = \frac{v^2}{2} \qquad \ln\left|\frac{x}{c}\right| = \frac{y^2}{2x^2}$$

$$|x| = |c|e^{\frac{y^2}{2x^2}}$$

$$x = c_1 e^{\frac{y^2}{2x^2}}$$

INTEGRATING FACTORS

1-52

Let $(2xy^2 - 3y^3)\,dx + (7 - 3xy^2)\,dy = 0$.

a. Show that this differential equation is not exact.

b. Find an integrating factor μ.

c. Solve the differential equation (implicit form).

****

$M = 2xy^2 - 3y^3$; $N = 7 - 3xy^2$.

$M_y = 4xy - 9y^2$; $N_x = -3y^2$.

1-54

Solve $\frac{dy}{dx} = \frac{(y+1)^4}{1 - 3(y+1)^3 x}$, $y(0) = 0$.

****

$(y+1)^4 dx + [3(y+1)^3 x - 1]\, dy = 0$

$M(x,y) = (y+1)^4$, $N(x,y) = 3(y+1)^3 - 1$, $M_y = 4(y+1)^3$, $N_x = 3(y+1)^3$

$\frac{N_x - M_y}{M} = \frac{-1}{y+1}$, so there is an integrating factor

$\mu(y) = \frac{1}{y+1}$. Now

$(y+1)^3 dx + \left[3(y+1)^2 x - \frac{1}{y+1}\right] dy = 0$ is an exact equation .

$\psi_x = (y+1)^3$

$\psi(x,y) = (y+1)^3 x + h(y)$

$\psi_y = 3(y+1)^2 x + h'(y) = 3(y+1)^2 x - \frac{1}{y+1}$,

so $h'(y) = \frac{-1}{y+1}$, $h(y) = -\ln(y+1) + C$.

Thus

$$\psi(x,y) = (y+1)^3 x - \ln(y+1) = C$$

is the general integral . By using the initial condition we have $C = 0$. Therefore the solution of the initial value problem is given by

$$(y+1)^3 x = \ln(y+1) \qquad \text{near } x = 0$$

1-53

Prove that $(y^2+y\cos x)dx + (3xy+2\sin x)dy = 0$ is not exact, find an integrating factor and then solve it.

****

$M(x,y) = y^2+y\cos x \qquad N(x,y) = 3xy+2\sin x$

$\frac{\partial M}{\partial y} = 2y+\cos x \neq \frac{\partial N}{\partial x} = 3y+2\cos x.$

$\therefore$ The equation is not exact.

$\frac{\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}}{N} = \frac{-y-\cos x}{3xy+2\sin x}$ is not independant of y and does not guarantee an integrating factor independent of y.

$\frac{\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}}{M} = \frac{y+\cos x}{y^2+y\cos x} = \frac{1}{y}$ is independent of x and guarantees an integrating factor independent of x.

The integrating factor is

$$\mu(y) = \exp\left(\int \frac{1}{y}\,dy\right) = \exp(\ln y) = y.$$

Multiply the original equation by $\mu(y)$ to obtain the exact equation

$$(y^3+y^2\cos x)\,dx + (3xy^2+2y\sin x)\,dy = 0.$$

Thus

$$F(x,y) = \int (y^3+y^2\cos x)\,dx + g(y)$$

$$= y^3x + y^2\sin x + g(y)$$

$$\frac{\partial F}{\partial y} = 3y^2x + 2y\sin x + g'(y)$$

But $\frac{\partial F}{\partial y} = 3xy^2+2y\sin x$. $\therefore g'(y) = 0$

Thus $F(x,y) = xy^3+y^2\sin x$, and the solution is

$$xy^3+y^2\sin x = C \quad \text{where } C \text{ is arbitrary.}$$

1-56

Solve: $y^3(y\,dx - x\,dy) = x(y\,dx + x\,dy)$.

****

SINCE $d\left(\frac{x}{y}\right) = \frac{y\,dx - x\,dy}{y^2}$ AND $d(xy) = y\,dx + x\,dy$, WE REWRITE:

$y^5 \frac{y\,dx - x\,dy}{y^2} = x(y\,dx + x\,dy)$ AND MULTIPLY BOTH SIDES BY $x^n y^k$, WHICH IS THE "INTEGRATING FACTOR."

$$\therefore x^n y^{k+5} \frac{y\,dx - x\,dy}{y^2} = x^{n+1} y^k (y\,dx + x\,dy). \qquad (\text{EQ. } 1.)$$

WE WANT $x^n y^{k+5}$ TO BE ANY POWER OF $\left(\frac{x}{y}\right)$ AT THE SAME TIME THAT WE WANT $x^{n+1} y^k$ TO BE ANY POWER OF (xy). THIS IS BECAUSE, e.g., $\int \left(\frac{x}{y}\right)^{10} d\left(\frac{x}{y}\right) = \frac{\left(\frac{x}{y}\right)^{11}}{11}$ OR $\int (xy)^{13} d(xy) = \frac{(xy)^{14}}{14}$.

NOW IF $k+5 = -n$, $x^n y^{k+5}$ BECOMES $x^n y^{-n}$ OR $\left(\frac{x}{y}\right)^n$

AND IF $n+1 = k$, $x^{n+1} y^k$ BECOMES $x^k y^k = (xy)^k$.

SOLVING THIS SYSTEM: $\left\{ \begin{matrix} k+5 = -n \\ n+1 = k \end{matrix} \right\}$ WE FIND $n = -3$, $k = -2$.

SUBSTITUTING INTO EQ. 1 AND REWRITING,

$$\frac{x^{-3}}{y^{-3}} \frac{y\,dx - x\,dy}{y^2} = x^{-2} y^{-2} (y\,dx + x\,dy)$$

OR $\left(\frac{x}{y}\right)^{-3} d\left(\frac{x}{y}\right) = (xy)^{-2} d(xy)$. INTEGRATING BOTH SIDES,

$$\frac{\left(\frac{x}{y}\right)^{-2}}{-2} = \frac{(xy)^{-1}}{-1} + C_1 \quad \text{OR} \quad \left(\frac{y}{x}\right)^2 = \frac{2}{xy} - 2C_1.$$

THIS CAN BE REWRITTEN $y^3 = x(2 + C_2 xy)$ WHERE $C_2 = -2C_1$.

1-55

Solve $y\,dx + (x^2 - x)\,dy = 0$ by using an integrating factor.

****

$M(x,y) = y$, $N(x,y) = x^2 - x$

$\frac{\partial M}{\partial y} = 1$, $\frac{\partial N}{\partial x} = 2x - 1$

$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$. Therefore the given equation is not exact.

But if we multiply the equation by the integrating factor $\frac{1}{x^2}$ it becomes

$$\frac{y}{x^2}\,dx + \left(1 - \frac{1}{x}\right)dy = 0$$

Now $M(x,y) = \frac{y}{x^2}$, $N(x,y) = 1 - \frac{1}{x}$

Therefore $\frac{\partial M}{\partial y} = \frac{1}{x^2}$, $\frac{\partial N}{\partial x} = \frac{1}{x^2}$

$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ and the equation is now exact.

Therefore the solution is given by $\varphi(x,y) = C'$, where

$$\varphi(x,y) = \int M\,dx + g(y) = \int \frac{y}{x^2}\,dx + g(y)$$

$$= -\frac{y}{x} + g(y)$$

$$\frac{\partial \varphi}{\partial y} = -\frac{1}{x} + g'(y) = N = 1 - \frac{1}{x}$$

$g'(y) = 1$ giving $g(y) = y + C_1$

and $\varphi(x,y) = -\frac{y}{x} + y + C_1$

Therefore the solution is $-\frac{y}{x} + y + C_1 = C'$

that is $-\frac{y}{x} + y = C$ $\qquad C = C' - C_1$

[Note: This diff. eq. may be solved also as a separable equation.]

1-58

Find an integrating factor of the form u(x) and find the general solution of $3\cos y + (x^{-2} - x\sin y)y' = 0$.

****

The equation $3\cos y\, u(x) + \left(\frac{1}{x^2} - x\sin y\right) u(x)\, y' = 0$ is exact if

$$\frac{\partial}{\partial x}\left[\left(\frac{1}{x^2} - x\sin y\right) u(x)\right] = \frac{\partial}{\partial y}\left[3\cos y\, u(x)\right]$$

$$\left(\frac{1}{x^2} - x\sin y\right) u'(x) + \left(-\frac{2}{x^3} - \sin y\right) u(x) = -3\sin y\, u(x)$$

$$\left(\frac{1}{x^2} - x\sin y\right) u'(x) = \left(\frac{2}{x^3} + \sin y - 3\sin y\right) u(x)$$

$$\frac{u'(x)}{u(x)} = \frac{\frac{2}{x^3} - 2\sin y}{\frac{1}{x^2} - x\sin y} = \frac{\frac{2}{x}\left(\frac{1}{x^2} - x\sin y\right)}{\frac{1}{x^2} - x\sin y} = \frac{2}{x}$$

$u'(x) = \frac{2}{x}\, u(x)$, so $u(x) = e^{\int 2/x} = e^{2\ln|x|} = e^{\ln x^2} = x^2$

Back to the top equation:

$3x^2\cos y + (1 - x^3\sin y)\, y' = 0$, which is exact.

The potential function ϕ satisfies

$$\frac{\partial \phi}{\partial x} = 3x^2\cos y \rightarrow \phi = \int 3x^2\cos y\, dx = x^3\cos y + C(y)$$

$$\frac{\partial \phi}{\partial y} = 1 - x^3\sin y \rightarrow \phi = \int 1 - x^3\sin y\, dy = y + x^3\cos y + C(x)$$

By comparison, $C(y) = y$ and $C(x) = 0$; $\phi = x^3\cos y + y$

The general solution is $\phi = c$, or

$x^3\cos y + y = c$.

1-57

Find an integrating factor of the form $\mu(x) = x^n$ for the equation given below. Then find the general solution.

$$(9 + 3x^6y^2)dx + (2x^7y - 2x^4)dy = 0$$

****

Method 1: $M = 9 + 3x^6y^2 \qquad M_y = 6x^6y$

$N = 2x^7y - 2x^4 \qquad N_x = 14x^6y - 8x^3$

$$\frac{M_y - N_x}{N} = \frac{6x^6y - (14x^6y - 8x^3)}{2x^7y - 2x^4} = \frac{-8x^6y + 8x^3}{2x^7y - 2x^4} = \frac{-8x^3(x^3y - 1)}{2x^4(x^3y - 1)}$$

$$= \frac{-4}{x}$$

$$\therefore \mu(x) = e^{\int \frac{-4}{x}dx} = e^{-4\ln|x|} = e^{\ln x^{-4}} = x^{-4}$$

Method 2: $x^n(9 + 3x^6y^2)dx + x^n(2x^7y - 2x^4)dy = x^n \cdot 0$

$$\underbrace{(9x^n + 3x^{n+6}y^2)}_{M^*}dx + \underbrace{(2x^{n+7}y - 2x^{n+4})}_{N^*}dy = 0$$

$$\frac{\partial M^*}{\partial y} = \frac{\partial N^*}{\partial x}$$

$$0 + 6x^{n+6}y = 2(n+7)x^{n+6}y - 2(n+4)x^{n+3}$$

$$\Rightarrow \begin{cases} 6 = 2(n+7) \\ 0 = -2(n+4) \end{cases} \qquad \therefore n = -4$$

$$\mu(x) = x^{-4}$$

General Solution:

$$x^{-4}(9 + 3x^6y^2)dx + x^{-4}(2x^7y - 2x^4)dy = 0$$

$$(9x^{-4} + 3x^2y^2)dx + (2x^3y - 2)dy = 0$$

(Note: this equation is exact.)

$$9x^{-4}dx + 3x^2y^2dx + 2x^3y\,dy - 2dy = 0$$

$$d(-3x^{-3}) + d(x^3y^2) - d(2y) = d(c)$$

$$-3x^{-3} + x^3y^2 - 2y = c$$

$$-3 + x^6y^2 - 2x^3y = cx^3$$

1-60

Solve the equation: $(4x^3y-3y^3+x^2y^2)dx + (3xy^2-2x^4)dy = 0$

****

Solution: Try an integration factor of the form $u(x,y) = x^n y^m$

$\Rightarrow \overbrace{x^n y^m (4x^3y-3y^3+x^2y^2)}^{M}dx + \overbrace{x^n y^m (3xy^2-2x^4)}^{N}dy = 0$

should now be exact. $\Rightarrow \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

$\Rightarrow \frac{\partial(4x^{n+3}y^{m+1} - 3x^n y^{m+3} + x^{n+2}y^{m+2})}{\partial y} = \frac{\partial(3x^{n+1}y^{m+2} - 2x^{n+4}y^m)}{\partial x}$

$\Rightarrow 4(m+1)x^{n+3}y^m - 3(m+3)x^n y^{m+2} + (m+2)x^{n+2}y^{m+1}$

$= 3(n+1)x^n y^{m+2} - 2(n+4)x^{n+3}y^m$

$\Rightarrow 4(m+1) = -2(n+4) \Rightarrow 2(m+1) = -n-4$

Also $-3(m+3) = 3(n+1) \Rightarrow -m-3 = n+1 \Rightarrow -m-4 = n$

Also $m+2 = 0 \Rightarrow \underline{m=-2} \Rightarrow -(-2)-4 = n \Rightarrow \underline{n=-2}$

$\Rightarrow u = x^{-2}y^{-2} \Rightarrow (4xy^{-1} - 3x^{-2}y + 1)dx + (3x^{-1} - 2x^2y^{-2})dy = 0$

is exact. So $F = \int (4xy^{-1} - 3x^{-2}y + 1)\partial x + T(y)$

$= 2x^2y^{-1} + 3x^{-1}y + x + T(y) = C \Rightarrow \frac{\partial F}{\partial y} = -2x^2y^{-2} + 3x^{-1} + T'(y)$

$= 3x^{-1} - 2x^2y^{-2} \Rightarrow T'(y) = 0$

$\Rightarrow 2x^2y^{-1} + 3x^{-1}y + x = C$ is the solution!

1-59

Find an integrating factor of the form $\mu(x,y) = x^m y^n$ and solve the following differential equation:

$$(2yx^{-1})\,dx + (y^2x^{-2} - 1)\,dy = 0$$

****

Suppose that $\mu(x,y) = x^m y^n$. Then multiplication gives

$$(2y^{n+1}x^{m-1})\,dx + (y^{n+2}x^{m-2} - x^m y^n)\,dy = 0.$$

Now $\dfrac{\partial M}{\partial y} = \dfrac{\partial}{\partial y}\left[2y^{n+1}x^{m-1}\right] = 2(n+1)x^{m-1}y^n$

and $\dfrac{\partial N}{\partial x} = \dfrac{\partial}{\partial x}\left[y^{n+2}x^{m-2} - x^m y^n\right]$

$= (m-2)\,y^{n+2}x^{m-3} - m\,x^{m-1}y^n.$

To be exact, we must have $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$.

Thus $2(n+1)x^{m-1}y^n = (m-2)\,y^{n+2}x^{m-3} - m\,x^{m-1}y^n.$

Equating coefficients gives $(m-2) = 0$ and $2(n+1) = -m$

So $m = 2$ and $n = -2$. Thus $\mu(x,y) = x^2y^{-2}$ and

$2y^{-1}x\,dx + (1 - x^2y^{-2})\,dy = 0$ is exact.

Now $F(x,y) = \int 2y^{-1}x\,dx + g(y) = x^2y^{-1} + g(y)$

Thus $\dfrac{\partial F}{\partial y} = -x^2y^{-2} + g'(y) = 1 - x^2y^{-2} \Rightarrow g(y) = y + C$

General Solution: $\underline{\underline{\dfrac{x^2}{2} + x\sin y - y^2 = C}}$

1-62

Solve the initial value problem

$$y' = \frac{3x - 2y^2}{2xy} \quad ; \quad y(1) = -1.$$

****

With y in the denominator it's not linear, so cross-multiply: $\frac{dy}{dx} = \frac{3x-2y^2}{2xy}$ yields

$2xy\,dy = (3x-2y^2)\,dx$, or $\underbrace{(2y^2-3x)}_{M}dx + \underbrace{2xy}_{N}dy = 0$

$\frac{\partial M}{\partial y} = 4y \neq 2y = \frac{\partial N}{\partial x}$ is not exact.

Integrating factor: $\frac{\partial M/\partial y - \partial N/\partial x}{N} = \frac{2y}{2xy} = \frac{1}{x}$

so an integrating factor $\mu = \mu(x)$ exists which depends only on x; it is a solution to $\mu' = \frac{1}{x}\cdot\mu$; the $\frac{1}{x}$ factor is $\frac{\partial M/\partial y - \partial N/\partial x}{N}$.

Thus $\frac{d\mu}{\mu} = \frac{dx}{x}$, and $\ln\mu = \ln x$ or $\mu = x$ works.

Multiplying, $\underbrace{(2y^2x - 3x^2)}_{M}dx + \underbrace{2x^2y}_{N}dy = 0.$

$\frac{\partial M}{\partial y} = 4yx = \frac{\partial N}{\partial x}$ now.

$\psi(x,y) = \int^{x}(2y^2x - 3x^2)\,dx = y^2x^2 - x^3 + h(y)$

$\frac{\partial \psi}{\partial y} = 2yx^2 + h'(y)$ must equal $N = 2x^2y$; O.K. without $h(y)$, and $y^2x^2 - x^3 = C$

Initial values: $(-1)^2 1^2 - 1^3 = C = 0$, so $y^2x^2 = x^3$ and $y^2 = x$ $(x_0 \neq 0)$ giving $\underline{\underline{y = -\sqrt{x}}}$ $(y_0 < 0)$.

1-61

Which of the following differential equations has an integrating factor given by $\mu = x^{-2}$?

a) $\frac{dy}{dx} + \frac{y}{x} = 1$

b) $\frac{dy}{dx} + (\cot x)y = \cos x$

c) $x\frac{dy}{dx} + 3y = x^2$

d) $\frac{dy}{dx} - 2\frac{y}{x} = x^2 \sin x$

****

Each of the differential equations is first order linear of the form $\frac{dy}{dx} + P(x)y = Q(x)$.

This equation always has an integrating factor given by $\mu(x) = e^{\int P(x)dx}$.

Testing each of the given differential equations, we note the following:

a) $P(x) = \frac{1}{x}$ and $\int P(x)dx = \int \frac{dx}{x} = \ln x$

Thus $\mu(x) = e^{\ln x} = x$

b) $P(x) = \cot x$ and $\int P(x)dx = \int \frac{\cos x}{\sin x}dx = \ln \sin x$

Thus $\mu(x) = e^{\ln \sin x} = \sin x$

c) $P(x) = \frac{3}{x}$ and $\int P(x)dx = 3\int \frac{dx}{x} = 3\ln x = x^3$

d) $P(x) = -\frac{2}{x}$, $\int P(x)\,dx = -2\int \frac{dx}{x} = -2\ln x$

ans. $-2\ln x = -\ln x^2$

Thus $\mu(x) = e^{-\ln x^2} = \frac{1}{x^2}$

1-64

Solve the differential equation $\frac{dy}{dx} = \frac{y^2 - x}{2xy}$. (1)

* *

Rewrite the equation as

$$(x - y^2)\,dx + 2xy\,dy = 0 \qquad (2)$$

which is of the form

$$M(x,y)\,dx + N(x,y)\,dy = 0.$$

Here

$$M(x,y) = x - y^2, \quad N(x,y) = 2xy.$$

We have

$$\frac{\partial M}{\partial y} = -2y, \quad \frac{\partial N}{\partial x} = 2y.$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, equation (2) is not exact. Note however that

$$\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{1}{2xy}(-4y) = -\frac{2}{x}$$

is a function of x only. This suggests that the function

$$I(x) = e^{-\int \frac{2}{x}dx} = e^{-2\ln x} = \frac{1}{x^2}$$

is an integrating factor for equation (2).

Multiplying equation (2) by $(1/x^2)$, we obtain the equation

$$\left(\frac{1}{x} - \frac{y^2}{x^2}\right)dx + \frac{2y}{x}\,dy = 0. \qquad (3)$$

1-63

Solve the following differential equation by first finding an integrating factor μ.

$$(3y^2 - x^2)dx - 2xy\,dy = 0$$

****

Recall, if we have an exact differential equation $M\,dx + N\,dy = 0$, then $M_y = N_x$. Clearly that is not true here - but it is worth checking. We wish an integrating factor μ such that $\mu M\,dx + \mu N\,dy = 0$ is exact. There is an integrating factor $\mu = \mu(x)$ if $\frac{M_y - N_x}{N} = f(x)$, and $\frac{d\mu}{dx} = f(x)\mu$. There is an integrating factor $\mu = \mu(y)$ if $\frac{N_x - M_y}{M} = g(y)$, and $\frac{d\mu}{dy} = g(y)\mu$.

Here, $M_y = 6y$, $N_x = -2y$ $\therefore f(x) = \frac{M_y - N_x}{N} = \frac{6y - (-2y)}{-2xy} = \frac{-4}{x}$

and $\frac{d\mu}{dx} = \left(\frac{-4}{x}\right)\mu \Rightarrow \int \frac{d\mu}{\mu} = \int -\frac{4}{x}\,dx$

$\Rightarrow \ln\mu = -4\ln x + \overset{\ln c_1}{c} = \ln(x^{-4}\cdot c_1)$

$\Rightarrow \mu = c_1 x^{-4}$ $\quad c_1 = 1$ without any loss of generality $\therefore \underline{\mu = x^{-4}}$. Multiplying by μ yields: $\underbrace{(3x^{-4}y^2 - x^{-2})}_{M^*}dx \underbrace{- 2x^{-3}y}_{N^*}\,dy = 0$

Note as a check: $\frac{\partial M^*}{\partial y} = 6x^{-4}y = \frac{\partial N^*}{\partial x}$.

Hence, there is a function $f(x,y)$ such that

$df = M^*dx + N^*dy = 0$ $\quad \frac{\partial f}{\partial x} = M^* = 3x^{-4}y^2 - x^{-2}$

$\overset{\parallel}{\frac{\partial f}{\partial x}}$ $\quad \overset{\parallel}{\frac{\partial f}{\partial y}}$ $\quad \therefore f = -x^{-3}y^2 + x^{-1} + C(y)$ $(*_1)$

But $-2x^{-3}y = N^* = \frac{\partial f}{\partial y} \overset{(*_1)}{=} -2x^{-3}y + C'(y)$

$\Rightarrow 0 = C'(y) \Rightarrow C(y) = K_1$; Since $df = 0 \Rightarrow f = -x^{-3}y^2 + x^{-1} + K_1 = K_2$ $(\Rightarrow f = K_2)$

Hence, $\underline{-x^{-3}y^2 + x^{-1} = K}$ $(K = K_2 - K_1)$

1-65

Find the general solution of the equation

$$e^x(1+x)dx + (ye^y - xe^x)dy = 0 \qquad (1)$$

****

This equation is neither separable, nor homogeneous, nor linear – nor exact. So we try to use integrating factor. First we may assume that the factor is a function of x alone, $\mu = \mu(x)$; to see if

$$\frac{\partial}{\partial y}(\mu(x)M) = \frac{\partial}{\partial x}(\mu(x)N)$$

holds for some $\mu(x)$, i.e. if the equation

$$0 = \mu'(x)(ye^y - xe^x) - \mu(x)e^x(1+x) \qquad (2)$$

can be solved for some $\mu(x)$. Our attempt fails because we got three variables x, y and μ in (2). Then we try $\mu = \mu(y)$, a function of y alone. Again we hope that

$$\frac{\partial}{\partial y}(\mu(y)M) = \frac{\partial}{\partial x}(\mu(y)N)$$

i.e. $$\mu'(y)e^x(1+x) = -\mu(y)e^x(1+x)$$

or $$\mu'(y) = -\mu(y)$$

Now we can solve $\mu(y)$, $\mu(y) = e^{-y}$. This is our desired integrating factor.

Letting $\bar{M}(x,y) = \frac{1}{x} - \frac{y^2}{x^2}$, $\bar{N}(x,y) = \frac{2y}{x}$

we see that

$$\frac{\partial \bar{M}}{\partial y} = -\frac{2y}{x^2}, \quad \frac{\partial \bar{N}}{\partial x} = -\frac{2y}{x^2}.$$

Therefore equation (3) is exact. This means that we can find a function $f(x,y)$ such that

$$\frac{\partial f}{\partial x} = \bar{M}(x,y) = \frac{1}{x} - \frac{y^2}{x^2} \tag{4}$$

and

$$\frac{\partial f}{\partial y} = \bar{N}(x,y) = \frac{2y}{x}. \tag{5}$$

Integrating equation (4) with respect to x, holding y fixed, we find

$$f(x,y) = \ln x + \frac{y^2}{x} + c(y) \tag{6}$$

where c is a function of y.

From equation (6), we have

$$\frac{\partial f}{\partial y} = \frac{2y}{x} + \frac{dc}{dy}.$$

In view of (5), we obtain

$$\frac{2y}{x} + \frac{dc}{dy} = \frac{2y}{x}$$

$$\Rightarrow \quad \frac{dc}{dy} = 0 \quad \Rightarrow \quad c(y) = \text{const.}$$

Therefore, from (6), the solution is:

$$\ln x + \frac{y^2}{x} = \text{const.}$$

1-66 LINEAR

Solve the following first order differential equations by multiplying suitable integrating factors.

(a) $t\dfrac{dy}{dt}+2y=t^3$, $y(1)=0$;

(b) $\dfrac{dy}{dx}-y=2e^x$, $y(0)=2$.

****

(a) The integrating factor is $e^{\int \frac{2}{t}dt} = e^{2\ln t} = t^2$.
Hence we can write the equation as

$$\frac{d}{dt}(t^2 y) = t^4$$

$$t^2 y(t) = \frac{t^5}{5} + c.$$

Since $y(1)=0$, therefore $c=-\frac{1}{5}$. The final solution

$$y(t) = \frac{t^3}{5} - \frac{1}{5t^2}.$$

(b) The integrating factor is $e^{-\int dx} = e^{-x}$. Multiply by e^x on both sides and group the first two terms

$$\frac{d}{dx}(e^x y) = 2.$$

Upon integration

$$e^{-x} y = 2x + c.$$

Knowing $y(0)=2$, we have $c=2$. The solution is

$$y(x) = 2xe^x + 2e^x.$$

Multiply e^{-y} to both side of equation (1), we obtain an exact equation

$$e^{x-y}(1+x)\,dx + (y - x e^{x-y})\,dy = 0 \quad (3)$$

The solution is of the form

$$f(x, y) = c$$

with
$$\frac{\partial f}{\partial x} = e^{-x-y}(1+x) \quad (4)$$

$$\frac{\partial f}{\partial y} = (y - x e^{x-y}) \quad (5)$$

Integrate (4) with respect to x

$$f(x, y) = x e^{x-y} + g(y) \quad (6)$$

To determine $g(y)$, differentiate (6) with respect to y

$$\frac{\partial f}{\partial y} = -x e^{x-y} + g'(y) \quad (7)$$

Compare (7) with (5), we find $g'(y) = y$, then $g(y) = \frac{1}{2} y^2$. Therefore by (6), the general solution of (3) is
$$x e^{x-y} + \frac{1}{2} y^2 = c$$

This is also the general solution of equation (1).

$P = 2x$

$\int P\,dx = x^2$

$e^{\int P\,dx} = e^{x^2}$

$y e^{\int P\,dx} = \int Q e^{\int P\,dx}\,dx$ is the general solution.

$$\int Q e^{\int P\,dx}\,dx = \int x e^{x^2}\,dx = \frac{1}{2} e^{x^2}$$

The general solution is

$$y e^{x^2} = \frac{1}{2} e^{x^2} + \frac{C}{2}$$

or $2y = Ce^{-x^2} + 1$

1-69

Solve $xy' + 2y = 4x^2$, $y(2) = 4$.

****

Rewrite as $y' + \frac{2}{x} y = 4x$.

The integrating factor is $e^{\int \frac{2}{x} dx} = e^{2 \ln x}$

$= x^2$.

Multiply by this factor to obtain

$$x^2 y' + 2xy = 4x^3.$$

Thus $(x^2 y)' = 4x^3$.

1-67

Find the general solution of the equation

$$xdy/dx + (\ln x)y = x/[(1+x^2)\cdot e^{\frac{1}{2}(\ln x)^2}]$$

****

Writing the equation in the form

$$\frac{dy}{dx} + \frac{\ln x}{x} y = \frac{1}{(1+x^2)e^{\frac{1}{2}(\ln x)^2}} \quad (1)$$

we find that the equation is linear. Multiply (1) by

$e^{\int \frac{\ln x}{x} dx} = e^{\frac{1}{2}(\ln x)^2}$:

$$e^{\frac{1}{2}(\ln x)^2}\frac{dy}{dx} + e^{\frac{1}{2}(\ln x)^2}\frac{\ln x}{x} y = \frac{1}{1+x^2}$$

i.e. $$\frac{d}{dx}\left[e^{\frac{1}{2}(\ln x)^2} y \right] = \frac{1}{1+x^2}$$

Integrate $$e^{\frac{1}{2}(\ln x)^2} y = \arctan x + C$$

Hence $$y = e^{-\frac{1}{2}(\ln x)^2}\arctan x + C$$

(The rule to solve a linear equation

$$\frac{dy}{dx} + P(x)y = Q(x)$$

is multiply by $e^{\int P(x)dx}$, then integrate.)

1-68

Find a general solution: y' = x-2xy

****

$\frac{dy}{dx} + 2xy = x$ is of the form $\frac{dy}{dx} + Py = Q$

1-71

Solve the intial-value problem $y' - 4y = 3x e^{4x}$, $y(0) = 1$

****

An integrating factor is $\int -4\,dx = e^{-4x}$

Therefore $e^{-4x}\left(\frac{dy}{dx} - 4y\right) = 3x e^{-4x} e^{4x}$

$$e^{-4x}\left(\frac{dy}{dx} - 4y\right) = 3x$$

Integrating we obtain $y e^{-4x} = \frac{3}{2}x^2 + C$

The initial condition gives $1 \cdot e^{-4(0)} = \frac{3}{2}(0)^2 + C$

that is $C = 1$

Therefore $y e^{-4x} = \frac{3}{2}x^2 + 1$

or $y = e^{4x}\left(\frac{3}{2}x^2 + 1\right)$ is the solution of the initial-value problem.

1-72

Solve the equation: $y' - x^2y + 3xy - x^4 + 2x^2 - 1 = 0$

****

Solution: $(1-x^2)y' + 3xy = x^4 - 2x^2 + 1 = (x^2-1)^2$

$$\Rightarrow y' + \frac{3x}{1-x^2}y = \frac{(1-x^2)^2}{1-x^2}$$

$$\Rightarrow y' + \underbrace{\frac{3x}{1-x^2}}_{P}y = \underbrace{1-x^2}_{Q}$$

Integrate to obtain

$$x^2 y = x^4 + c.$$

Since $y(2) = 4$, $2^2 \cdot 4 = 2^4 + c$.

Therefore $c = 0$ and thus

$$x^2 y = x^4 \quad \text{or} \quad y = x^2.$$

1-70

Find the general solution of $xy' + 2y = e^{3x}$; for which values of x is the solution valid?

****

The equation is linear, and should be rewritten as $y' + \frac{2}{x} y = \frac{e^{3x}}{x}$. An integrating factor is $e^{\int \frac{2}{x}} = e^{2 \ln|x|} = e^{\ln x^2} = x^2$. Multiplying by x^2,

$x^2 y' + 2xy = xe^{3x}$, or $(x^2 y)' = xe^{3x}$. Then

$x^2 y = \int xe^{3x}\,dx$. Integrating by parts with $\begin{cases} u = x, du = dx \\ dv = e^{3x}\,dx \\ v = \frac{e^{3x}}{3} \end{cases}$

$$x^2 y = x\frac{e^{3x}}{3} - \int 1 \frac{e^{3x}}{3}\,dx$$

$$x^2 y = \frac{xe^{3x}}{3} - \frac{e^{3x}}{9} + C. \quad \text{Finally,}$$

$$y(x) = \frac{e^{3x}}{3x} - \frac{e^{3x}}{9x^2} + \frac{C}{x^2}, \text{ which is valid for } x \neq 0.$$

1-74

Determine the solution of the linear IVP

$$y' + \frac{(1+x)}{x} y = e^{-x}, \quad x > 0$$

$$y(1) = 0.$$

****

This equation is already in standard form

$$y' + P(x)y = Q(x)$$

The integrating factor is $\mu(x) = e^{\int P(x)\,dx}$, so here $\mu(x) = e^{\int \frac{1+x}{x}\,dx} = e^{\ln x + x} = xe^x$.

Multiply through by the integrating factor to obtain

$$xe^x\left(y' + \frac{1+x}{x} y\right) = xe^x e^{-x} = x.$$

The left side is now an exact derivative

$$\frac{d}{dx}(xe^x y) = x$$

Integrate both sides:

$$xe^x y = \frac{1}{2}x^2 + c.$$

The general solution is $y = \frac{1}{2}xe^{-x} + \frac{c}{x}e^{-x}$

Apply the initial condition $y(1) = 0$.

$$\Rightarrow \quad 0 = \frac{1}{2}e^{-1} + ce^{-1} \Rightarrow c = -\frac{1}{2}$$

$$\boxed{y = \frac{1}{2}\left(x - \frac{1}{x}\right)e^{-x}}$$

$$\Rightarrow vy = \int vQ\,dx + C \quad \text{where } v = \exp\left[\int \frac{3x\,dx}{1-x^2}\right]$$

$$= \exp\left[\frac{-3}{2}\ln(1-x^2)\right] = (1-x^2)^{-3/2}$$

$$\Rightarrow (1-x^2)^{-3/2} y = \int (1-x^2)^{-3/2}(1-x^2)\,dx + C$$

$$= \int (1-x^2)^{-1/2}\,dx + C = \sin^{-1}x + C$$

$$\Rightarrow y = (1-x^2)^{3/2}(\sin^{-1}x + C)$$

1-73

Solve the initial value problem

$$y' = \frac{1 - 2xy}{x^2}\ ;\ y(1) = 2$$

****

$y' = \frac{1}{x^2} - \frac{2}{x}y$; rewriting,

$y' + \frac{2}{x}y = \frac{1}{x^2}$. This is linear.

$\mu(x) = e^{\int \frac{2}{x}dx} = e^{2\ln|x|} = x^2$ is the integrating factor, producing $x^2y' + 2xy = 1$.

This is $(x^2y)' = 1$, giving $x^2y = \int 1\,dx = x + c$

Initial values: $1^2 \cdot 2 = 1 + c$ gives $c = 1$, and

$x^2y = x + 1$, Ans.: $y = \frac{x+1}{x^2}$, or $y = \frac{1}{x} + \frac{1}{x^2}$.

1-76

Solve $xy' + 5y = 2x + 3$, $y(1) = 3$.

****

This is a linear equation. Put it in standard form:

$$y' + \frac{5}{x}y = 2 + \frac{3}{x}$$

The integrating factor is $e^{\int f}$. Here, $f = \frac{5}{x}$ so the integrating factor is $e^{5\ln x} = x^5$. Multiply:

$$x^5y' + 5x^4 = 2x^5 + 3x^4$$

$$(x^5y)' = 2x^5 + 3x^4$$ Integrate:

$$x^5y = \frac{1}{3}x^6 + \frac{3}{5}x^5 + C$$

$$\boxed{y = \frac{1}{3}x + \frac{3}{5} + Cx^{-5}}$$

Now $y(1) = \frac{1}{3} + \frac{3}{5} + C = 3 \Rightarrow$

$$C = \frac{45}{15} - \frac{5}{15} - \frac{9}{15} = \frac{31}{15}$$

So $$\boxed{y = \frac{1}{3}x + \frac{3}{5} + \frac{31}{15}x^{-5}}$$

1-75

Consider the ordinary differential equation $\frac{dx}{x} - \frac{dy}{y} = 0$.

a. Determine whether or not the D. E. is linear [justify].

b. Determine whether or not the D. E. is separable [justify].

c. Determine whether or not the D. E. is exact [justify].

d. Determine whether or not the D. E. is homogeneous [justify].

e. Solve the initial value problem $\frac{dx}{x} - \frac{dy}{y} = 0;\ y(1) = 2$.

****

a. Multiply by $-y/dx$ to obtain $\frac{dy}{dx} - \frac{1}{x}y = 0$. The D. E. is linear.

b. Since $M = M(x) = 1/x$ and $N = N(y) = -1/y$ are functions of only x and only y, respectively, the D. E. is separable.

c. $M_y = \frac{\partial M}{\partial y} = \frac{\partial}{\partial y}\left(\frac{1}{x}\right) = 0$; $N_x = \frac{\partial N}{\partial x} = \frac{\partial}{\partial x}\left(-\frac{1}{y}\right) = 0$. The D.E. is exact.

d. From part "a.", $\frac{dy}{dx} = y/x = f(x,y)$ and f is homogeneous of degree zero; the D.E. is homogeneous.

e. Any of the solution methods associated with linear, separable, exact, or homogeneous may be used. For example, the separable method gives: $\int_1^x \frac{dt}{t} + \int_2^y -\frac{ds}{s} = 0$

$$\ln x - \ln 1 - \ln y + \ln 2 = 0$$

$$\ln\left(\frac{2x}{y}\right) = 0 \text{ or } \frac{2x}{y} = e^0 = 1$$

or $y = 2x$.

1-78

Solve the differential equation $x\frac{dy}{dx} - y = x^2 + 2$.

* * * * * * * * * * * * * * * * * * * *

Note that this is a linear equation for $y(x)$. Dividing both sides of the equation by x, we have

$$\frac{dy}{dx} - \frac{1}{x}y = x + \frac{2}{x}. \qquad (*)$$

Equation (*) is of the form

$$\frac{dy}{dx} + p(x)y = q(x),$$

which has an integrating factor $e^{\int p(x)dx}$.

Here $p(x) = -\frac{1}{x}$, $q(x) = x + \frac{2}{x}$.

Therefore the integrating factor is:

$$e^{-\int \frac{1}{x}dx} = e^{-\ln x} = \frac{1}{x}.$$

Multiplying both sides of equation (*) by $1/x$, we obtain

$$\frac{1}{x}\frac{dy}{dx} - \frac{1}{x^2}y = 1 + \frac{2}{x^2}.$$

This last equation can be written as

$$\frac{d}{dx}\left(\frac{1}{x}y\right) = 1 + \frac{2}{x^2},$$

which on integration gives

$$\frac{1}{x}y = x - \frac{2}{x} + c \qquad (c = \text{const.})$$

Hence the solution is given by

$$y = x^2 - 2 + cx.$$

1-77

Find the general solution: $(1 - 2xy - x^4)dx + x^2 dy = 0$

****

$$1 - 2xy - x^4 + x^2 \frac{dy}{dx} = 0$$

$$x^2 \frac{dy}{dx} - 2xy = x^4 - 1$$

$$\frac{dy}{dx} - \frac{2}{x} y = x^2 - \frac{1}{x^2}$$

∴ THIS IS A LINEAR EQUATION, SO WE COMPUTE THE INTEGRATING FACTOR AS FOLLOWS:

$$\mu(x) = e^{\int \frac{-2}{x} dx} = e^{-2\ln|x|} = e^{\ln x^{-2}} = x^{-2}$$

$$\Rightarrow \quad x^{-2}\left(\frac{dy}{dx} - 2x^{-1}y\right) = x^{-2}\left(x^2 - x^{-2}\right)$$

$$x^{-2}\frac{dy}{dx} - 2x^{-3}y = x^0 - x^{-4} = 1 - x^{-4}$$

NOTE: THE LEFT-HAND SIDE IS THE RESULT OBTAINED BY DIFFERENTIATING $x^{-2}y$ BY THE PRODUCT RULE, SO

$$\frac{d}{dx}\left(x^{-2}y\right) = 1 - x^{-4}$$

$$d\left(x^{-2}y\right) = \left(1 - x^{-4}\right)dx$$

$$\int d\left(x^{-2}y\right) = \int \left(1 - x^{-4}\right)dx$$

$$x^{-2}y = x - \left(\frac{x^{-3}}{-3}\right) + C$$

$$\frac{y}{x^2} = x + \frac{1}{3x^3} + C$$

$$y = x^3 + \frac{1}{3x} + Cx^2$$

OR: $3xy = 3x^4 + 1 + C_1 x^3$

WHERE $C_1 = 3C$

1-80

(*) $x^2 \frac{dy}{dx} + xy = 1$. $y =$ (A) $(\ln x)/x + C/x$

(B) $\ln x + C/x$

(C) $C/x - (\ln x)/x$

(D) $C/x - \ln x$

(E) $\ln x + C$

****

We first solve the equation

$$(H) \quad x^2 \frac{dy}{dx} + xy = 0 ,$$

by separating the variables x and y. To that end, we factor

$$x\left(x \frac{dy}{dx} + y\right) = 0$$

Since x is not identically zero, we must have

$$x \frac{dy}{dx} + y = 0$$

Thus,

$$x\,dy = -y\,dx$$

and the variables may be separated

$$\frac{1}{y}\,dy = -\frac{1}{x}\,dx$$

Integrating both sides,

$$\ln y + \underbrace{C_1}_{\text{constant}} = -\ln x + \underbrace{C_2}_{\text{constant}}$$

$$\ln y = -\ln x + \underbrace{C_2 - C_1}$$

$$(C_2 - C_1)\ln e = \ln e^{\underbrace{C_2 - C_1}_{\text{a constant that we call } C.}}$$

$$= -\ln x + \ln C$$

$$= \ln \frac{C}{x} .$$

Therefore,

$y = \frac{C}{x}$ solves the equation (H).

1-79

Solve the following differential equation for the solution that passes through the point (1,3) and indicate the (maximum) solution interval for this solution.

$$x\frac{dy}{dx} = -2y + 5x^3$$

****

$x\frac{dy}{dx} + 2y = 5x^3$ (Note, for later: $\frac{dy}{dx} = -\frac{2}{x}y + 5x^2 = g(x,y)$)

$\Rightarrow \frac{dy}{dx} + \frac{2}{x}y = 5x^2$ $(*_1)$

For this linear differential equation, an integrating factor $\mu = e^{\int \frac{2}{x}dx}$ exists.

$\mu = e^{2\ln x} = e^{\ln x^2} = x^2$. Hence we have, from $(*_1)$,

$$x^2\left[\frac{dy}{dx} + \frac{2}{x}y\right] = 5x^4$$

$$\therefore \frac{d}{dx}\left[x^2 \cdot y\right] = 5x^4$$

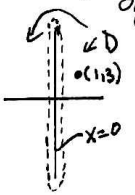
$\Rightarrow x^2y = x^5 + C \Rightarrow y = x^3 + \frac{C}{x^2}$; $(1,3) \Rightarrow 3 = 1 + \frac{C}{1}$

$\Rightarrow C = 2$. $\therefore y = x^3 + \frac{2}{x^2}$ is the unique solution through (1,3).

The region, D, in which we have a unique solution is

$D = \begin{cases} g \text{ is cont.} & (g \text{ is cont. where } x \neq 0 \\ \frac{\partial g}{\partial y} \text{ is cont.} & (-\frac{\partial g}{\partial y} = -\frac{2}{x} \text{ is also cont. where } x \neq 0 \end{cases}$

$\therefore D = R^2 - \{x=0\}$



By D.E. theory, our solution can be extended until it "passes out of D". Our solution can be extended from (1,3) to the left until $x=0$. To the right, the graph of the solution stays in D for all large x. Hence, the maximum solution interval is $(0, +\infty)$.

1-81

Find a general solution to $xy' - y = (x^3)(1 - x^2)^{\frac{1}{2}}$ valid for $x = 1/3$.

****

IF A DIFFERENTIAL EQUATION CAN BE WRITTEN IN THE FORM $y' + P(x)y = Q(x)$, i.e., A FIRST ORDER, LINEAR EQUATION, THEN THE GENERAL SOLUTION CAN BE WRITTEN

$$y = e^{-\int P(x)dx} \int e^{\int P(x)dx} Q(x)\,dx + c\,e^{-\int P(x)dx}$$

WHERE $e^{\int P(x)dx}$ IS THE "INTEGRATING FACTOR" FOR THIS FORM.

WE REWRITE THE GIVEN EQ. AS $y' - \frac{1}{x}y = x^2(1-x^2)^{1/2}$.

THEN $P(x) = -\frac{1}{x}$, $Q(x) = x^2(1-x^2)^{1/2}$, AND $\int P(x)\,dx = \int -\frac{1}{x}\,dx = -\ln|x|$

SINCE WE ARE INTERESTED IN $x = \frac{1}{3}$, WE CAN CHOOSE $x > 0$, IN WHICH CASE

$\int P(x)\,dx = -\ln x$. $e^{\int P(x)dx} = e^{-\ln x}$, AND SINCE $-\ln x = \ln x^{-1}$,

$e^{-\ln x} = e^{\ln x^{-1}} = x^{-1}$. ($A^{\log_A B} = B$ IS AN IDENTITY.)

$e^{-\int P(x)dx} = e^{\ln x} = x$.

$$\int e^{\int P(x)dx} Q(x)\,dx = \int x^{-1}(x^2)(1-x^2)^{1/2}\,dx = \int x(1-x^2)^{1/2}\,dx$$

$$\int x(1-x^2)^{1/2}\,dx = -\frac{1}{2}\int (1-x^2)^{1/2}(-2x\,dx) = -\frac{1}{2}\,\frac{(1-x^2)^{3/2}}{\frac{3}{2}}$$

$$= -\frac{1}{3}(1-x^2)^{3/2}.$$

$$\therefore\quad y = x\left[-\frac{1}{3}(1-x^2)^{3/2}\right] + cx = cx - \frac{x}{3}(1-x^2)^{3/2}.$$

THIS SOLUTION IS VALID FOR ALL POSITIVE x, INCLUDING $x = \frac{1}{3}$. IT IS "GENERAL" BECAUSE IT CONTAINS ONE ESSENTIAL (UNREMOVABLE) ARBITRARY CONSTANT.

We presume that the solution to the original equation (*) will be

$$y = \frac{C(x)}{x} = C(x)\frac{1}{x},$$

where the constant (parameter) C that appears in the solution of (H) has been replaced by a function (a variable parameter) $C(x)$. The method is called "variation of parameter" since the parameter now varies with x.

Now we need the product rule to differentiate $y = C(x)\cdot\frac{1}{x}$.

$$\frac{dy}{dx} = C'(x)\cdot\frac{1}{x} + C(x)\left(-\frac{1}{x^2}\right).$$

Substituting $y = C(x)\cdot\frac{1}{x}$ and $\frac{dy}{dx}$ into the equaton (*), we obtain

$$x^2\left(\frac{C'(x)}{x} - \frac{C(x)}{x^2}\right) + x\left(\frac{C(x)}{x}\right) = 1,$$

which reduces to

$$x\,C'(x) = 1$$

That is, a differential equation for $C(x)$. We have

$$C'(x) = \frac{1}{x},$$

Integrating,

$$C(x) = \ln x + \text{constant}.$$

Thus, the solution of (*) is

$$y = \frac{C(x)}{x} = \frac{\ln x}{x} + \frac{\text{Constant}}{x}$$

This is the answer (A).

1-83

Solve $\frac{dy}{dx} + \frac{2}{x} y = x$

****

This is a 1st Order Linear Eq. of the form

$\frac{dy}{dx} + p(x) y = g(x)$ where $p(x) = \frac{2}{x}$ and $g(x) = x$.

The general solution is given by $y = \frac{1}{\mu}\left[\int \mu g \, dx + c\right]$

where $\mu = e^{\int p(x)\,dx}$. So

$\mu = e^{\int \frac{2}{x} dx} = e^{2\ln|x|} = e^{\ln|x|^2} = e^{\ln x^2} = x^2$ and

$$y = \frac{1}{x^2}\left[\int x^2 \cdot x \, dx + c\right] = \frac{1}{x^2}\left[\int x^3 dx + c\right]$$

$$= \frac{1}{x^2}\left[\frac{1}{4}x^4 + c\right]$$

The solution is therefore

$$y = \frac{1}{4}x^2 + \frac{c}{x^2}.$$

1-84

Solve the differential equation

$$y' = \sec x - y \tan x$$

****

This is linear. $y' + (\tan x) y = \sec x$.

Integrating factor $\mu = \exp\left(\int \tan x \, dx\right)$

1-82

Solve $y' + xy = x$ with initial condition $y(0) = 0$. Sketch the solution.

****

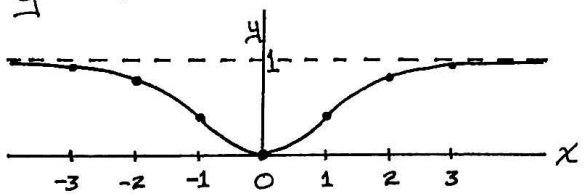
The integrating factor is $e^{\int x\,dx} = e^{\frac{1}{2}x^2}$.

Multiplying both sides of the equation by it, you get $e^{\frac{1}{2}x^2} \cdot y' + xe^{\frac{1}{2}x^2} \cdot y = xe^{\frac{1}{2}x^2}$.

Next, recognize the left side as an exact derivative: $\frac{d}{dx}\left[ye^{\frac{1}{2}x^2}\right]$. Hence you can integrate both sides, obtaining

$$ye^{\frac{1}{2}x^2} = \int xe^{\frac{1}{2}x^2}dx + C$$
$$= e^{\frac{1}{2}x^2} + C$$

Substituting $x = 0$ & $y = 0$ gives $C = -1$, so $y = 1 - e^{-\frac{1}{2}x^2}$ is the solution.



The above sketch shows that the solution is an upside-down normal (or "bell") curve.

This gives $y = -Ke^{-x} + C$. Now let $C_1 = -K$ and $C_2 = C$ to get $y = C_1 e^{-x} + C_2$ as the general solution.

1-86

Solve $y'' - y' - 2 = 0$

****

This second order diff. equation can be reduced to an equation of first order by the substitution

$$z = y'$$

The given equation then reduces to $\frac{dz}{dx} - z = 2$, which is linear and of first order.

Multiplying both sides by the integrating factor $e^{\int -1\,dx} = e^{-x}$, we get

$$e^{-x}\left(\frac{dz}{dx} - z\right) = 2e^{-x}$$

Integrating we obtain

$$ze^{-x} = C_1 + \int 2e^{-x}\,dx$$

$$ze^{-x} = C_1 - 2e^{-x}$$

that is $$z = C_1 e^{x} - 2$$

But $z = \frac{dy}{dx}$

Therefore $$\frac{dy}{dx} = C_1 e^{x} - 2$$

Integrating, we obtain

$$y = C_1 e^{x} - 2x + C_2$$, which is the solution of the given second order equation and contains two arbitrary constants C_1 and C_2.

$\mu = e^{\int \tan x\,dx} = e^{-\ln|\cos x|} = \sec x$ (since constants in exponents and $\pm$ signs do not matter for integrating factors).

$(y \sec x)' = y' \sec x + y \sec x \tan x = \sec^2 x$, so

$$y \sec x = \int \sec^2 x\,dx = \tan x + c$$

$$y = \tan x \cos x + c \cdot \cos x = \underline{\underline{\sin x + c \cdot \cos x}}$$

REDUCTION TO FIRST ORDER

1-85

Solve $y'' + y' = 0$.

****

Since this equation does not contain y explicitly we can reduce the order by substituting $v = y'$. Then $v' = y''$ and $v' + v = 0$ is the DE.

This is a 1st order linear homogeneous DE (i.e. has the form $y' + p(x)y = 0$ whose general solution is $y = c e^{-\int p(x)dx}$). Hence $v = k e^{-\int dx} = k e^{-x}$ is the solution.

Re substituting gives $y' = k e^{-x}$. Integrating

1-88

Solve the initial value problem

$$\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} = 0, \qquad \begin{matrix} y(1) = 0, \\ y'(1) = 1. \end{matrix}$$

****

Here the dependent variable is missing, so the equation is reducible to first order.

Set $\frac{dy}{dx} = v$, $\frac{d^2y}{dx^2} = \frac{dv}{dx}$

The equation becomes $\frac{dv}{dx} + \frac{1}{x}v = 0$,

with the initial condition $y'(1) = v(1) = 1$.

The equation is linear for $v(x)$, so with the integrating factor $\mu(x) = e^{\int \frac{dx}{x}} = e^{\ln x} = x$,

$$x\left(\frac{dv}{dx} + \frac{1}{x}v\right) = 0 \Rightarrow \frac{d}{dx}(xv) = 0.$$

Thus $xv = c_1 \Rightarrow v = \frac{c_1}{x}$. $v(1) = c_1 = 1$.

$$v = \frac{dy}{dx} = \frac{1}{x}$$

Integrate $y = \ln x + c_2$

$$y(1) = \ln 1 + c_2 = 0 \Rightarrow c_2 = 0.$$

$$\boxed{y = \ln x}$$

1-87

Use the substitution $p = \frac{dy}{dx}$ to reduce the order of the second order equation $y'' = \frac{1}{2}(y'^2+1)/y$ and then find its general solution.

****

Let $p = \frac{dy}{dx}$. Then $y'' = \frac{dp}{dx} = \frac{dp}{dy}\frac{dy}{dx} = p\frac{dp}{dy}$.

Substitute these in the equation, to obtain

$$p\frac{dp}{dy} = \frac{1}{2}(p^2+1)/y.$$

Seperate variables and integrate

$$\int \frac{2p}{p^2+1}dp = \int \frac{1}{y}dy$$

$$\ln(p^2+1) = \ln|y| + C$$

$$p^2+1 = e^C|y| = A|y|, \text{ where } A = e^C > 0.$$

$$p^2 = A|y|-1$$

$$p = \pm\sqrt{Ay-1} \quad (\text{assume } y>0)$$

i.e. $$\frac{dy}{dx} = \pm\sqrt{Ay-1}$$

Seperate variables and integrate

$$\int \frac{dy}{\sqrt{Ay-1}} = \pm\int dx$$

$$\frac{2}{A}\sqrt{Ay-1} = \pm(x+B)$$

Square both sides and multiply by $A^2/4$

$$Ay-1 = \frac{A^2}{4}(x+B)^2$$

$$y = \frac{1}{A} + \frac{A}{4}(x+B)^2, \quad A, B \text{ arbitrary}, A>0.$$

The case $y<0$ can be treated in a similar way.

1-90

Solve completely the following second order non-linear equation:

$$y''(t) = 2\,y(t)\,y'(t)$$

with the following initial conditions: $y(0) = -1$ and $y'(0) = 0$.

(Hint: Put $v(t) = y'(t)$, get a first order equation in v.)

****

Set $V(t) = y'(t)$, then by chain rule

$$y''(t) = \frac{dV}{dt} = \frac{dV}{dy}\frac{dy}{dt} = V\frac{dV}{dy}.$$

Put into the given equation

$$V\frac{dV}{dy} = 2yV.$$

Cancelling and separation of variable

$$dV = 2y\,dy$$

and when integrated

$$V = y^2 + C_1$$

where C_1 is an arbitrary constant. Use the initial conditions

$$V(0) = y(0)^2 + C_1$$

which gives $C_1 = -1$. Therefore we have

$$\frac{dy}{dt} = y^2 - 1$$

and separation of variables again

$$\frac{dy}{y^2-1} = dx.$$

Integrate on both sides

$$\ln|y-1| - \ln|y+1| = 2x + C_2$$

where C_2 is another arbitrary constant.

1-89

Find the general solution of the equation

$$x^2y'' + xy' = 1$$

****

This is a second order equation with y missing. Let $p = y'$ to reduce the order:

$$x^2 p' + xp = 1 \qquad (1).$$

which is a linear equation. We may use the formula for the solution of the linear equation to find

$$p(x) = e^{-\int \frac{1}{x}\,dx}\left(\int \frac{1}{x^2} e^{\int \frac{1}{x}\,dx} + c\right)$$

while in most cases (if we forget the formula), we may do like this: Divide (1) by x (to make the left hand side a derivative of something), we obtain

$$xp' + p = \frac{1}{x}, \quad \text{i.e.} \quad \frac{d}{dx}(xp) = \frac{1}{x}$$

Integrate both side $\quad xp = \ln x + c_1$ or

$$y'(x) = p(x) = \frac{\ln x}{x} + \frac{c_1}{x}$$

Integrate again

$$y(x) = \frac{1}{2}(\ln x)^2 + c_1 \ln x + c_2.$$

This is the desired solution.

(Note that the general solution of a second order equation always contains two arbitrary constant c_1 and c_2).

1-91

Solve the following differential equation :

$$x(1 - x)y'' + (1 - 2x)y' = 0. \qquad (1)$$

* *

Since the dependent variable y does not appear explicitly in the differential equation, we can reduce this equation to a first order equation by the substitution

$$y' = u \quad \text{and therefore} \quad y'' = u'.$$

Equation (1) then reduces to

$$x(1-x)u' + (1-2x)u = 0.$$

Separating the variables u and x and integrating, we have

$$\int \frac{du}{u} = \int \frac{1 - 2x}{x(x-1)}\,dx. \qquad (2)$$

The integral on the right hand side can be evaluated using the method of partial fractions.

Write $$\frac{1 - 2x}{x(x-1)} = \frac{A}{x} + \frac{B}{x-1}.$$

Multiply both sides of this last equation by $x(x-1)$. Then

$$1 - 2x = A(x-1) + Bx.$$

Now using $x = 0$, we find $A = -1$
and with $x = 1$, we have $B = -1$.

When we enforce the initial condition: $y(0) = -1$ we get into trouble since $\ln|y+1|$ is not defined when $y = -1$. So there does not exist a solution to the differential equation of the form

$$\ln|y-1| - \ln|y+1| = 2x + C_2$$

for any choice of C_2.

However we see that there exists a trivial solution

$$y = -1$$

which satisfies the differential equation together with the initial conditions.

1-92

Find the solution of the given differential equation satisfying the given initial condition.

$$\frac{d^2y}{dx^2} = 2 \cdot \frac{dy}{dx} \cdot y^3 \; ; \quad y(0) = 1, \; y'(0) = \frac{1}{2} \; .$$

****

$y'' = 2y'\cdot y^3$, let $v = \frac{dy}{dx}$, $v' = \frac{dv}{dx} = \frac{dv}{dy}\frac{dy}{dx} = \frac{dv}{dy}\cdot v$ (y'')

$\Rightarrow v\frac{dv}{dy} = 2vy^3$

$\Rightarrow dv = 2y^3dy \Rightarrow v = \frac{1}{2}y^4 + C$ but, imposing the initial conditions $\frac{1}{2} = y'(0) = v(0) = \frac{1}{2}\left[y(0)\right]^4 + C$

$\therefore \frac{1}{2} = \frac{1}{2}(1) + C \Rightarrow C = 0$

Hence, $v = \frac{1}{2}y^4$

$\therefore \frac{dy}{dx} = \frac{1}{2}y^4$

$\int y^{-4}\,dy = \int \frac{1}{2}\,dx \Rightarrow -\frac{1}{3}y^{-3} = \frac{1}{2}x + C_1$ Again, using the initial conditions: $-\frac{1}{3}(y(0))^{-3} = 0 + C_1$ (y'')

$\Rightarrow C_1 = -\frac{1}{3}$ $\therefore -\frac{1}{3}y^{-3} = \frac{1}{2}x - \frac{1}{3}$

$\Rightarrow y^{-3} = -\frac{3}{2}x + 1 = \frac{2-3x}{2}$ $\therefore \frac{1}{y^3} = \frac{2-3x}{2}$

$\Rightarrow y^3 = \frac{2}{2-3x}$ or $y = \sqrt[3]{\frac{2}{2-3x}}$

Thus $$\int \frac{1-2x}{x(x-1)}\,dx = -\int \left(\frac{1}{x} + \frac{1}{x-1}\right) dx$$

$$= -[\ln x + \ln(x-1)] = -\ln[x(x-1)].$$

Accordingly, equation (2) gives

$$\ln u = -\ln[x(x-1)] + \ln c_1$$

where c_1 is an arbitrary constant.

Simplifying the last equation, we may write

$$u = \frac{c_1}{x(x-1)}.$$

Therefore

$$\frac{dy}{dx} = \frac{c_1}{x(x-1)}$$

which can be integrated to give

$$y = c_1 \int \frac{1}{x(x-1)}\,dx + c_2,$$

c_2 being another arbitrary constant.

The integral on the right hand side can be calculated by the method of partial fractions.

The final solution is then given by

$$y = c_1 \ln\left[\frac{x-1}{x}\right] + c_2.$$

1-94

Solve the nonlinear equation $y\frac{d^2y}{dx^2} - \left(\frac{dy}{dx}\right)^2 = 0$. (1)

* *

Since the independent variable x does not appear explicitly in this equation, we can reduce it to a first order equation using the substitution

$$u = dy/dx.$$

Then $$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dy}\left(\frac{dy}{dx}\right)\frac{dy}{dx} = \left(\frac{du}{dy}\right)u.$$

Equation (1) now becomes

$$yu\frac{du}{dy} - u^2 = 0$$

so that

$$u\left[y\frac{du}{dy} - u\right] = 0.$$

Therefore

either $u = 0 \Rightarrow y' = 0 \Rightarrow y = \text{const.}$

or $$y\frac{du}{dy} - u = 0. \qquad (2)$$

Note that equation (2) is separable and has the solution

$$\ln u = \ln y + \ln c_1 \qquad (3)$$

where c_1 is an arbitrary constant.

1-93

Solve the initial value problem

$$y(y')^2 + (y^2 - 8(y')^2)y'' = 0; \quad y(0) = 2,\ y'(0) = -1.$$

****

Setting $y' = v$ and $y'' = \frac{dv}{dx} = \frac{dy}{dx}\frac{dv}{dy} = v\frac{dv}{dy}$, this becomes first order: $yv^2 + (y^2 - 8v^2)v\frac{dv}{dy} = 0$, or $\underbrace{yv^2}_{M}\,dy + \underbrace{(y^2v - 8v^3)}_{N}\,dv = 0$. $\frac{\partial M}{\partial v} = 2yv = \frac{\partial N}{\partial y}$ so this is exact. $\psi(y,v) = \int^y yv^2\,dy = \frac{1}{2}y^2v^2 + h(v)$. $\frac{\partial \psi}{\partial v} = y^2v + h'(v)$ must equal $y^2v - 8v^3$, so $h'(v) = -8v^3$, and $h(v) = -2v^4$ will do.

$\frac{1}{2}y^2v^2 - 2v^4 = C$. Initial values are $y_0 = 2$, $v_0 = -1$, so $\frac{1}{2}\cdot 2^2\cdot(-1)^2 - 2(-1)^4 = C = 0$. So: $\frac{1}{4}y^2v^2 = v^4$ gives $\frac{1}{4}y^2 = v^2$ (since $v \neq 0$), which gives $v = -\frac{1}{2}y$ (since y_0 and v_0 have opposite sign). $\frac{dy}{dx} = -\frac{1}{2}y$ is constant coefficient linear with solution $y = ce^{-x/2}$. Initial values are $x_0 = 0$, $y_0 = 2$, so $2 = ce^0$ and $\underline{y = 2e^{-x/2}}$.

$4y^3v^2 - y = c$. Initial values:

$y_0 = 2$, $v_0 = \frac{1}{4}$, so $4 \cdot 2^3 \left(\frac{1}{4}\right)^2 - 2 = c = 0$, so

$4y^3v^2 - y = 0$ and $4y^2v^2 = 1$ (since $y \neq 0$).

Further, $2yv = 1$ (since y_0, v_0 are both positive).

$2y\frac{dy}{dx} = 1$, so $2y\,dy = dx$ and $\int 2y\,dy = \int dx$.

$y^2 = x + c$. Initial values: $2^2 = 4 + c$, so $c = 0$,

$y^2 = x$ and $y = +\sqrt{x}$ (since y_0 is positive).

PICARD

1-96

For $y' = 1 + xy$, $y(0) = 1$ use Picard's method to find y_1, y_2, y_3 of the approximation.

****

For $y' = f(x,y)$, $y(0) = y_0$ we form the approximations

$$y_n = y_0 + \int_0^x f(t, y_{n-1})\,dt, \quad n = 1, 2, \cdots$$

$$y_1 = y_0 + \int_0^x (1 + t \cdot 1)\,dt = 1 + \left(t + \frac{t^2}{2}\right)\Big|_0^x = 1 + x + \frac{x^2}{2}$$

$$y_2 = 1 + \int_0^x \left[1 + t\left(1 + t + \frac{t^2}{2}\right)\right] dt = 1 + \left(t + \frac{t^2}{2} + \frac{t^3}{3} + \frac{t^4}{8}\right)\Big|_0^x =$$

$$= 1 + x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{8}$$

$$y_3 = 1 + \int_0^x \left[1 + t\left(1 + t + \frac{t^2}{2} + \frac{t^3}{3} + \frac{t^4}{8}\right)\right] dt =$$

$$= 1 + x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{8} + \frac{x^5}{15} + \frac{x^6}{48}.$$

Equation (3) may be expressed as

$$u = c_1 y.$$

Since $u = dy/dx$, it follows that

$$\frac{dy}{dx} = c_1 y$$

which is also a separable equation. This has the solution

$$\ln y = c_1 x + c_2$$

so that

$$y = \bar{c}_2 e^{c_1 x}$$

where $\bar{c}_2 = e^{c_2}$ is an arbitrary constant.

1-95

Solve the initial value problem

$$8y^3y'' + 12y^2(y')^2 - 1 = 0; \quad y(4) = 2, \quad y'(4) = \tfrac{1}{4}.$$

****

No x in the equation. substitute $v = y'$ and

$y'' = \frac{dv}{dx} = \frac{dy}{dx}\frac{dv}{dy} = v\frac{dv}{dy}$.

$$8y^3 v\frac{dv}{dy} + 12y^2v^2 - 1 = 0 = \underbrace{8y^3v}_{M}dv + \underbrace{(12y^2v^2 - 1)}_{N}dy$$

$\frac{\partial M}{\partial y} = 24y^2v = \frac{\partial N}{\partial v}$, so this equation is exact.

$\psi(y,v) = \int^v 8y^3v\,dv = 4y^3v^2 + h(y)$

$\frac{\partial \psi}{\partial y} = 12y^2v^2 + h'(y)$ must equal $12y^2v^2 - 1$, so

$h'(y) = -1$, and $h(y) = -y$ works.

1-98

Use Picard's method to solve the initial value problem given by

$$y' - 3x^2 y = 0, \quad y(0) = 1.$$

* *

The differential equation is of the form

$$y' = f(x, y) = 3x^2 y$$

with $y(0) = y_0 = 1$.

In Picard's method, the successive approximations to the solution are found as follows.

$$y_1(x) = y_0 + \int_0^x f(x, y_0)\, dx$$

$$= 1 + \int_0^x 3x^2 (1)\, dx = 1 + x^3,$$

$$y_2(x) = y_0 + \int_0^x f(x, y_1(x))\, dx$$

$$= 1 + \int_0^x 3x^2 (1 + x^3)\, dx$$

$$= 1 + 3\int_0^x (x^2 + x^5)\, dx = 1 + x^3 + \frac{x^6}{2},$$

$$y_3(x) = y_0 + \int_0^x f(x, y_2(x))\, dx$$

$$= 1 + \int_0^x 3x^2 \left(1 + x^3 + \frac{x^6}{2}\right) dx$$

1-97

Find the Picard iterates for the intial-value problem

$$\frac{dy}{dt} = ty \quad , \quad y(0) = 1$$

and show that they converge to the solution of the problem.

****

To obtain Picard iterates, the given diff. equation with the initial condition may be equivalently written as

$$y(t) = 1 + \int_0^t s\,y(s)\,ds$$

Therefore

$$y_0(t) = 1$$

$$y_1(t) = 1 + \int_0^t s\,y_0(s)\,ds = 1 + \int_0^t s\,ds = 1 + \frac{t^2}{2}$$

$$y_2(t) = 1 + \int_0^t s\,y_1(s)\,ds = 1 + \int_0^t s\left(1 + \frac{s^2}{2}\right)ds = 1 + \frac{1}{1!}\left(\frac{t^2}{2}\right) + \frac{1}{2!}\left(\frac{t^2}{2}\right)^2$$

$$y_3(t) = 1 + \int_0^t s\,y_2(s)\,ds = 1 + \int_0^t s\left(1 + \frac{1}{1!}\left(\frac{s^2}{2}\right) + \frac{1}{2!}\left(\frac{s^2}{2}\right)^2\right)ds$$

$$= 1 + \frac{1}{1!}\left(\frac{t^2}{2}\right) + \frac{1}{2!}\left(\frac{t^2}{2}\right)^2 + \frac{1}{3!}\left(\frac{t^2}{2}\right)^3$$

Therefore,

$$y_n(t) = 1 + \int_0^t s\,y_{n-1}(s)\,ds$$

$$= 1 + \int_0^t s\left[1 + \frac{1}{1!}\left(\frac{s^2}{2}\right) + \frac{1}{2!}\left(\frac{s^2}{2}\right)^2 + \frac{1}{3!}\left(\frac{s^2}{2}\right)^3 + \cdots\cdots + \frac{1}{(n-1)!}\left(\frac{s^2}{2}\right)^{n-1}\right]ds$$

$$= 1 + \frac{1}{1!}\left(\frac{t^2}{2}\right) + \frac{1}{2!}\left(\frac{t^2}{2}\right)^2 + \frac{1}{3!}\left(\frac{t^2}{2}\right)^3 + \cdots\cdots + \frac{1}{n!}\left(\frac{t^2}{2}\right)^n$$

As $n \to \infty$, $y_n(t) \to e^{t^2/2}$; that is $\lim_{n\to\infty} y_n(t) = e^{t^2/2}$

Therefore Picard iterates converge to $e^{t^2/2}$, which is the solution of the problem.

Check:

The solution of the problem may be obtained by treating the diff. equation as separable.

$\frac{dy}{y} = t\,dt$ on integration gives $\ln|y| = C + \frac{t^2}{2}$. Applying the initial condition $\ln|1| = C + \frac{(0)^2}{2}$ gives $C = 0$.

Therefore $\ln|y| = \frac{t^2}{2}$, that is $y = e^{t^2/2}$.

1-99

Solve the following initial value problem

$$\frac{dy}{dx} = x^2 - y, \quad y(0) = 0$$

by Picard's method of successive approximations.

****

The differential equation can be written in the integral form

$$y(x) = 0 + \int_0^x (s^2 - y(s))\, ds$$

We start with $y_0(x) = 0$, the successive approximations:

$$y_1(x) = \int_0^x (s^2 - 0)\, ds = \frac{1}{3}x^3$$

$$y_2(x) = \int_0^x (s^2 - \frac{s^3}{3})\, ds = \frac{x^3}{3} - \frac{x^4}{3\cdot 4}$$

$$y_3(x) = \int_0^x (s^2 - \frac{s^3}{3} + \frac{s^4}{3\cdot 4})\, ds = \frac{x^3}{3} - \frac{x^4}{3\cdot 4} + \frac{x^5}{3\cdot 4\cdot 5}$$

$$\vdots$$

$$y_n(x) = \frac{x^3}{3} - \frac{x^4}{3\cdot 4} + \cdots + \frac{(-1)^{n+1} x^{n+2}}{3\cdot 4 \cdots (n+2)}$$

The solution to the initial value problem:

$$y(x) = \lim_{n\to\infty} y_n(x) = -2 \sum_{n=3}^{\infty} \frac{(-x)^n}{n!}$$

$$= -2 \sum_{n=0}^{\infty} \frac{(-x)^n}{n!} + 2 - 2x + x^2$$

$$= 2 - 2x + x^2 - 2e^{-x}.$$

This agrees with the solution obtained by the method of integrating factor.

i.e. $y_3(x) = 1 + x^3 + \frac{x^6}{2} + \frac{x^9}{6}$,

$$y_4(x) = y_0 + \int_0^x f(x, y_3(x))\,dx$$

$$= 1 + \int_0^x 3x^2\left(1 + x^3 + \frac{x^6}{2} + \frac{x^9}{6}\right)dx$$

$$= 1 + x^3 + \frac{x^6}{2} + \frac{x^9}{3.2} + \frac{x^{12}}{4.3.2}$$

and so on. Observe that a definite pattern is emerging. We may therefore write the solution as

$$y(x) = 1 + x^3 + \frac{x^6}{2!} + \frac{x^9}{3!} + \frac{x^{12}}{4!} + \cdots + \frac{x^{3n}}{n!} + \cdots$$

Note that the equation $y' = 3x^2y$, subject to the initial condition $y(0) = 1$, has an exact solution $y = e^{x^3}$, which can indeed be expanded into the power series obtained above by Picard's method.

1-101

Use the following differential equation for parts a) - c):

$$x\frac{dy}{dx} + y = 4$$

a) Solve by the method of separation of variables.
b) Solve by the exact method.
c) Solve by the method of integrating factors.

****

a) $x\frac{dy}{dx} = 4-y \Rightarrow x\,dy = (4-y)\,dx$

$\Rightarrow \frac{dy}{4-y} = \frac{dx}{x} \Rightarrow -\ln|4-y| = \ln x + \ln c$

or $\frac{1}{4-y} = cx$

b) by exact method, first put into correct form of $M\,dx + N\,dy = 0$:

$(4-y)\,dx - x\,dy = 0$ $\quad M = 4-y \quad M_y = -1$
$\quad N = -x \quad N_x = -1$

so the differential equation is exact

$f(x,y) = \int (4-y)\,dx = 4x - xy + g(y)$

$\frac{\partial f}{\partial y} = -x + g'(y) = -x \ (= N)$

$\Rightarrow g'(y) = 0 \Rightarrow g(y) = c$

so the solution is $4x - xy + c = c_1$
or $4x - xy = c_2$

c) first put into integrating factor form of $\frac{dy}{dx} + P(x)y = q(x)$

$\frac{dy}{dx} + \frac{1}{x}y = \frac{4}{x}$ $\qquad \mu = e^{\int \frac{1}{x}dx} =$

$e^{\ln x} = x$

MISCELLANEOUS TECHNIQUES

1-100

Solve $y' + y = xy^3$

****

Because of the factor y^3 in the RHS, the equation is not linear. Equation of the form $y' + P(x)y = Q(x)y^m$ is the Bernoulli differential equation which reduces to a linear equation in z when we change to the dependent variable z given by the substitution $z = y^{1-m}$

Then $\frac{1}{y^3}\frac{dy}{dx} + \frac{1}{y^2} = x$ $\qquad z = y^{1-3} = y^{-2}$, $\frac{dz}{dx} = -2y^{-3}\frac{dy}{dx}$

$-\frac{1}{2}\frac{dz}{dx} + z = x$

$\frac{dz}{dx} - 2z = -2x$, which is linear.

Multiply by the integrating factor $e^{\int -2dx} = e^{-2x}$

$e^{-2x}\left(\frac{dz}{dx} - 2z\right) = -2x e^{-2x}$

Integrating we obtain

$$z e^{-2x} = C - 2\int x e^{-2x} dx$$
$$= C - 2\left[-\frac{x}{2}e^{-2x} - \frac{e^{-2x}}{(-2)^2}\right]$$
$$= C + x e^{-2x} + \frac{1}{2}e^{-2x}$$

Therefore $z = Ce^{2x} + x + \frac{1}{2}$

and the solution is given by

$$\frac{1}{y^2} = Ce^{2x} + x + \tfrac{1}{2}$$

or $y = \pm\left(Ce^{2x} + x + \tfrac{1}{2}\right)^{-\frac{1}{2}}$

Therefore $df = 0$ ∫

$$f(x,y) = c_2$$

or $x^2y + 9y + c_1 = c_2$

$\Rightarrow$ $x^2y + 9y = c$ is solution.

Solution #3: This differential equation can also be put in the form $\frac{dy}{dx} + p(x)y = q(x)$.

$$\frac{dy}{dx} + \frac{2x}{x^2+9}y = 0 \qquad \mu = e^{\int \frac{2x}{x^2+9}\,dx}$$

$$= e^{\ln|x^2+9|} = x^2+9$$

so the solution is:

$$y(x^2+9) = \int 0 \cdot (x^2+9)\,dx = \int 0 = c$$

$$y(x^2+9) = c$$

1-103

By making appropriate substitutions transform the following differential equation into a homogeneous differential equation: (x-2)dx + 4(x+y-1)dy=0

****

Let $x-2=0$ and $x+y-1=0$

$x = 2$

$y = 1-x$

$y = -1$

The solution will be

$$xy = \int x \cdot \frac{4}{x}\,dx = \int 4\,dx = 4x + C$$

or $\underline{xy = 4x + C}$

Note that the answers in a), b) & c) are all the same, although in a slightly different form.

1-102

Solve the following differential equation: $2xydx + (x^2 + 9)\,dy = 0$.

****

Soln #1: separate variables: $\frac{2x}{x^2+9}\,dx + \frac{dy}{y} = 0$

$$\ln|x^2+9| + \ln|y| = \ln C$$

$$\underline{y(x^2+9) = C}$$

Soln #2: exact: $M = 2xy \quad N = x^2+9$

$M_y = 2x \quad N_x = 2x$

so the differential equation is exact & there is a fn. $f(x,y)$ such that

$$\frac{\partial f}{\partial x} = M = 2xy \quad \& \quad \frac{\partial f}{\partial y} = N = x^2+9$$

$$f = \int 2xy\,dx = x^2y + g(y)$$

$$\frac{\partial f}{\partial y} = x^2 + g'(y) = x^2 + 9 \Rightarrow g'(y) = 9$$

$$\Rightarrow g(y) = 9y + C_1$$

so $f = x^2y + 9y + C_1$

or $y = \frac{2x}{1+c_1x^2}$ (Note $c_1 = 2c$)

Also note that this problem is homogeneous of degree 2 & can be solved by letting $y = ux$, $dy = u\,dx + x\,du$.

Substituting yields:

$$x^2(u\,dx + x\,du) + (x\cdot ux - u^2x^2)\,dx = 0$$

divide all terms by x^2:

$$u\,dx + x\,du + (u-u^2)\,dx = 0$$
$$(2u-u^2)\,dx + x\,du = 0$$

or separating the variables:

$$\frac{dx}{x} + \frac{du}{2u-u^2} = 0$$

$$\frac{dx}{x} + \left(\frac{A}{u} + \frac{B}{2-u}\right)du = 0 \qquad \left[2A - Au + Bu = 1 \Rightarrow A = 1/2,\ B = 1/2\right]$$

$$\frac{dx}{x} + \frac{1}{2}\left[\frac{1}{u} + \frac{1}{2-u}\right]du = 0$$

$$\ln|x| + \frac{1}{2}\left[\ln|u| - \ln|2-u|\right] = \ln c$$

$$\Rightarrow \quad x\cdot\left(\frac{u}{2-u}\right)^{1/2} = c$$

resubstituting: $u = y/x$ gives

$$x\left(\frac{y/x}{2-y/x}\right)^{1/2} = c$$

$$\Rightarrow \quad x\left(\frac{y}{2x-y}\right)^{1/2} = c$$

The substitution is: $x = u + 2 \quad dx = du$
$y = v - 1 \quad dy = dv$

The equation becomes

$$(u+2-2)\,du + 4(u+2+v-1-1)\,dv = 0$$

$$u\,du + 4(u+v)\,dv = 0$$

This is homogeneous.

1-104

Solve the following differential equation: $x^2\frac{dy}{dx} + xy = y^2$.

****

$$x^2\frac{dy}{dx} + xy = y^2 \Rightarrow \frac{dy}{dx} + \frac{1}{x}y = \frac{y^2}{x^2}$$

This is a Bernoulli differential equation with $w = y^{1-2} = y^{-1}$

$$\frac{dw}{dx} = -y^{-2}\frac{dy}{dx} \Rightarrow -y^2\frac{dw}{dx} = \frac{dy}{dx}$$

substituting: $\frac{dw}{dx} - \frac{1}{x}w = -\frac{1}{x^2}$

Now we have an integrating factor problem:

$$\mu = e^{\int -\frac{1}{x}dx} = e^{-\ln x} = e^{\ln x^{-1}} = x^{-1} = \frac{1}{x}$$

&

$$\frac{w}{x} = \int \frac{1}{x}\cdot\frac{-1}{x^2}\,dx = \int -x^{-3}\,dx$$

$$\frac{w}{x} = \frac{-x^{-2}}{-2} + C \Rightarrow w = \frac{1}{2x} + Cx$$

resubstituting: $y^{-1} = \frac{1}{2x} + Cx = \frac{1+2Cx^2}{2x}$

1-106

Solve the Bernouilli DE $y' = y - y^3$.

****

The Bernoulli DE $y' + P(x)y = Q(x)y^n$, $n \neq 0, 1$, may be reduced to a linear DE by the substitution $v = y^{1-n}$.

Here $n=3$, put $v = y^{1-3} = y^{-2}$

So $y = v^{-1/2}$, $y' = -\frac{1}{2}v^{-3/2}v'$

Substituting into the DE find

$$-\tfrac{1}{2}v^{-3/2}v' = v^{-1/2} - (v^{-1/2})^3 = v^{-1/2} - v^{-3/2}$$

Multiply through by $2v^{3/2}$ and get

$$v' + 2v = 2$$

Integrating factor $\mu(x) = e^{\int 2\,dx} = e^{2x}$

$$e^{2x}(v' + 2v) = 2e^{2x}$$

$$\Rightarrow \frac{d}{dx}(ve^{2x}) = 2e^{2x}$$

Integrate: $ve^{2x} = e^{2x} + C$

$$v(x) = 1 + ce^{-2x}$$

Since $y(x) = v^{-1/2}$

$$\boxed{y(x) = (1 + ce^{-2x})^{-1/2}}$$

1-105

Solve $\dfrac{dy}{dx} = \dfrac{x + 2y + 1}{2x + y - 1}$

****

Change to new variables given by the substitution $x = \bar{x} + h$, $y = \bar{y} + k$

Then $x + 2y + 1 = \bar{x} + h + 2(\bar{y} + k) + 1$
$= \bar{x} + 2\bar{y} + h + 2k + 1$

$2x + y - 1 = 2(\bar{x} + h) + (\bar{y} + k) - 1$
$= 2\bar{x} + \bar{y} + 2h + k - 1$

We put $h + 2k + 1 = 0$, $2h + k - 1 = 0$ giving us $h = 1$, $k = -1$

That is $x = \bar{x} + 1$, $y = \bar{y} - 1$ and $x + 2y + 1 = \bar{x} + 2\bar{y}$, $2x + y - 1 = 2\bar{x} + \bar{y}$. Also $\dfrac{dy}{dx} = \dfrac{dy}{d\bar{y}} \dfrac{d\bar{y}}{d\bar{x}} \dfrac{d\bar{x}}{dx} = (1)\left(\dfrac{d\bar{y}}{d\bar{x}}\right)(1) = \dfrac{d\bar{y}}{d\bar{x}}$

Therefore the given eq. reduces to the homogeneous eq.

$$\frac{d\bar{y}}{d\bar{x}} = \frac{\bar{x} + 2\bar{y}}{2\bar{x} + \bar{y}}.$$

Substitute $\bar{y} = v\bar{x}$, $\dfrac{d\bar{y}}{d\bar{x}} = v + \bar{x}\dfrac{dv}{d\bar{x}}$

Then $v + \bar{x}\dfrac{dv}{d\bar{x}} = \dfrac{\bar{x} + 2v\bar{x}}{2\bar{x} + v\bar{x}} = \dfrac{1 + 2v}{2 + v}$

$$\bar{x}\frac{dv}{d\bar{x}} = \frac{1 + 2v}{2 + v} - v = \frac{1 - v^2}{2 + v}$$

Integrating

$$\int \frac{(2 + v)}{1 - v^2}\, dv - \int \frac{1}{\bar{x}}\, d\bar{x} = C_1$$

$$\int \left[\frac{3/2}{1 - v} + \frac{1/2}{1 + v}\right] dv - \int \frac{1}{\bar{x}}\, d\bar{x} = C_1$$

$$-\tfrac{3}{2} \ln|1 - v| + \tfrac{1}{2} \ln|1 + v| - \ln|\bar{x}| = C_1$$

$$\ln\left|\frac{1 + v}{\bar{x}^2 (1 - v)^3}\right| = C_2 \quad (C_2 = 2C_1) \text{ giving } \frac{1 + v}{(1 - v)^3 \bar{x}^2} = C \qquad C > 0$$

$$(C = e^{C_2})$$

that is, $\dfrac{1 + \bar{y}/\bar{x}}{(1 - \bar{y}/\bar{x})^3 \bar{x}^2} = C$

$$\frac{\bar{x} + \bar{y}}{(\bar{x} - \bar{y})^3} = C$$

$$(x - 1 + y + 1) = C(x - 1 - y - 1)^3$$

Therefore the solution is $x + y = C(x - y - 2)^3$

1-108

Use the following differential equation for parts a) and b).

$$y' = \frac{x^2 + y^2}{xy}, \quad y(1) = 3$$

a) Use the Euler approximation method to approximate y(1.4). Use h = .2.

b) Solve the differential equation to find the true value of y(1.4).

****

a)

x_n	y_n	$y_{n+1} = y_n + .2\left(\frac{x_n^2 + y_n^2}{x_n y_n}\right)$
1	3	$3 + .2\left(\frac{1^2+3^2}{1\cdot 3}\right) = 3 + .2\left(\frac{10}{3}\right) = 11/3$
1.2	$11/3$	$11/3 + .2\left(\frac{(1.2)^2+(11/3)^2}{(1.2)(11/3)}\right) = 4.3432$
1.4	4.3432	

so $y(1.4) \approx 4.3432$

b) this is a homogeneous differential equation

let $y = ux$ $\quad dy = u\,dx + x\,du$

$$\frac{u\,dx + x\,du}{dx} = \frac{x^2 + u^2x^2}{x(ux)} = \frac{1+u^2}{u}$$

$$u^2\,dx + ux\,du = dx + u^2\,dx \Rightarrow ux\,du = dx$$

separate the variables: $u\,du = \frac{dx}{x}$

so $\frac{u^2}{2} = \ln x + C \Rightarrow \frac{y^2}{2x^2} = \ln x + C$

$y(1) = 3$: $\frac{3^2}{2\cdot 1^2} = \ln 1 + C \Rightarrow C = 9/2$

so $\frac{y^2}{2x^2} = \ln x + 9/2$

Now $y(1.4)$: $\frac{y^2}{2(1.4)^2} = \ln(1.4) + 9/2$

$\Rightarrow y = \pm 4.3542$

1-107

Solve the differential equation $y' - 2y = y^2$ when $y=3$ and $x=0$.

****

This is a Bernoulli type equation. Let $v = y^{1-2} = y^{-1} = \frac{1}{y}$. Then $\frac{dv}{dx} = -\frac{1}{y^2}\frac{dy}{dx}$.

Substitute into the differental equation to obtain $-y^2\frac{dv}{dx} - 2y = y^2$.

Thus $\frac{dv}{dx} = -\frac{2}{y} - 1$ or $\frac{dv}{dx} + 2v = -1$.

This is a linear equation with integrating factor of e^{2x}. Multiply both sides by this to obtain

$$e^{2x}\left(\frac{dv}{dx} + 2v\right) = -e^{2x}.$$

Thus $$\frac{d}{dx}\left(e^{2x}v\right) = -e^{2x}.$$

Integrate, to obtain $e^{2x}v = -\frac{1}{2}e^{2x} + c_1$.

Since $v = 1/y$, we obtain $e^{2x}\left(\frac{1}{y}\right) = -\frac{1}{2}e^{2x} + c_1$.

Now let $x=0$ and $y=3$. This gives $\frac{1}{3} = -\frac{1}{2} + c_1$. Hence $c_1 = 5/6$.

We can write $\frac{e^{2x}}{y} = -\frac{1}{2}e^{2x} + \frac{5}{6}$.

When this is solved for y, the solution can be written as

$$y = \frac{6e^{2x}}{5 - 3e^{2x}}.$$

1-110

Solve (implicitly) the differential equation

$$y' = \frac{y+2}{x+1} + \tan\left(\frac{y-2x}{x+1}\right).$$

[Hint: Let $x = X + h$ and $y = Y + k$ for suitable constants h and k.]

****

For $x = X+h$ and $y = Y+k$, $dx = dX$ and $dy = dY$.
Also $y' = dy/dx = dY/dX$. We obtain the D.E.

$$\frac{dY}{dX} = \frac{Y+k+2}{X+h+1} + \tan\left(\frac{Y+k-2X-2h}{X+h+1}\right).$$

Take $h+1=0$ and $k+2=0$, or $h=-1$ and $k=-2$.
The D.E. becomes

$$dY/dX = Y/X + \tan\left(\frac{Y-2X}{X}\right), \text{ or}$$

$$dY/dX = Y/X + \tan(Y/X - 2), \text{ a}$$

homogeneous D.E.

Let $Y = XV$. Then $Y' = V + XV'$.

$$V + XV' = V + \tan(V-2).$$

$$XV' = \tan(V-2)$$

$$\frac{dV}{\tan(V-2)} = \frac{dX}{X}$$

$$\frac{\cos(V-2)}{\sin(V-2)}\,dV = \frac{dX}{X}$$

$$\frac{d(\sin(V-2))}{\sin(V-2)} = \frac{dX}{X}$$

$$\ln|\sin(V-2)| = \ln|X| + C$$

$$|\sin(V-2)| = K|X|$$

1-109

(a) Find the solution of the homogeneous differential equation:

$$\frac{dy}{dx} = \frac{x - y}{x + y}$$

(b) Indicate the substitution $\begin{cases} \bar{X} = x - h \\ \bar{Y} = y - k \end{cases}$ (h, k to be determined)

which will transform $\frac{dy}{dx} = \frac{x - y + 4}{x + y + 2}$ into a homogeneous differential equation of the form in (a).

****

(a) $\frac{dy}{dx} = \frac{1 - y/x}{1 + y/x}$, let $v = y/x \Rightarrow y = xv$

$\Rightarrow y' = xv' + v$

$\Rightarrow xv' + v = \frac{1 - v}{1 + v} \Rightarrow xv' = \frac{1 - v}{1 + v} - v = \frac{1 - v - v(1+v)}{1 + v}$

$\therefore x\frac{dv}{dx} = \frac{1 - 2v - v^2}{1 + v} \Rightarrow \frac{1 + v}{1 - 2v - v^2}dv = \frac{dx}{x}$ Integrating:

$-\frac{1}{2}\int \frac{-2(1+v)\,dv}{1 - 2v - v^2} = \int \frac{dx}{x}$

$z = 1 - 2v - v^2$

$dz = (-2 - 2v)dv$

$dz = -2(1+v)dv$

$-\frac{1}{2}\int \frac{dz}{z} = \ln|x| + C$ " $\ln K$

$\Rightarrow -\frac{1}{2}\ln|1 - 2v - v^2| = \ln|x| + \ln K$

$\Rightarrow \ln|1 - 2v - v^2| = -2\ln|x| + \ln K_1$

$\ln|1 - 2v - v^2| + \ln x^2 + \ln K_2 = 0$

$\ln\left[|1 - 2v - v^2| \cdot x^2 \cdot K_2\right] = 0$

$\Rightarrow |1 - 2v - v^2| \cdot x^2 \cdot K_2 = 1$

$\left|1 - 2\frac{y}{x} - \frac{y^2}{x^2}\right| \cdot x^2 \cdot K_2 = 1$

$\Rightarrow |x^2 - 2yx - y^2| K_2 = 1$

$\Rightarrow \underline{x^2 - 2yx - y^2 = C}$

(b) $\begin{cases} X = x - h \Rightarrow x = X + h \\ Y = y - k \Rightarrow y = Y + k \end{cases}$

$\Rightarrow dx = dX$

$\Rightarrow dy = dY$

$\Rightarrow \frac{dy}{dx} = \frac{x - y + 4}{x + y + 2}$ becomes

$\frac{dY}{dX} = \frac{X + h - (Y + k) + 4}{X + h + (Y + k) + 2} = \frac{X - Y + (h - k + 4)}{X + Y + (h + k + 2)}$

of form (a) if:

$h - k + 4 = 0$

and $h + k + 2 = 0$, solving

$\Rightarrow h = -3,\ k = 1$

$\therefore X = x + 3,\ Y = y - 1$

will transform the differential equation into the form in (a).

used

1-112

Find a one parameter family of solutions for the differential equation $xy' + y - y^2x^3e^{2x} = 0$.

****

$$xy' + y = x^3e^{2x}y^2$$

$$y' + \frac{1}{x}y = x^2e^{2x}y^2$$ a Bernoulli equation

use the substitution $w = y^{1-n}$, where $n=2$

$$w = y^{1-2} = y^{-1} = \frac{1}{y} \qquad y = \frac{1}{w} = w^{-1}$$

$$\frac{dy}{dx} = -w^{-2}\frac{dw}{dx}$$

into the de: $-w^{-2}\frac{dw}{dx} + \frac{1}{x}w^{-1} = x^2e^{2x}(w^{-1})^2$

$$\frac{dw}{dx} - \frac{1}{x}w = -x^2e^{2x}$$

first order linear: $\mu(x) = e^{\int -\frac{1}{x}dx} = e^{-\ln x} = \frac{1}{x}$

$$\frac{1}{x}\frac{dw}{dx} - \frac{1}{x^2}w = -xe^{2x}$$

$$\frac{d}{dx}\left(\frac{1}{x}w\right) = -xe^{2x}$$

$$\frac{1}{x}w = \int -xe^{2x}\,dx$$

integration by parts: $u = -x$, $dv = e^{2x}dx$; $du = -dx$, $v = \frac{1}{2}e^{2x}$

$$\frac{1}{x}w = \frac{-x}{2}e^{2x} - \int -\frac{1}{2}e^{2x}\,dx$$

$$\frac{1}{x}w = \frac{-x}{2}e^{2x} + \frac{1}{4}e^{2x} + C$$

$$|\sin(\tfrac{y}{x}-2)| = K|x|$$
$$|\sin(\tfrac{y-2x}{x})| = K|x|$$
$$|\sin(\tfrac{y+2-2x-2}{x+1})| = K|x+1|$$
$$|\sin(\tfrac{y-2x}{x+1})| = K|x+1|.$$

1-111

Use an appropriate substitution to solve:

$$xy'' = y' + (y')^3$$

****

Let $u = y'$. Then $\frac{du}{dx} = y''$ and substitution gives

$$x\frac{du}{dx} - u = u^3, \quad \text{or} \quad \frac{du}{dx} - \frac{u}{x} = \frac{u^3}{x}.$$

This is a Bernoulli equation and we make the substitution $w = u^{-2}$, giving

$$\frac{dw}{dx} + \frac{2w}{x} = \frac{-2}{x}$$

This is a linear equation having x^2 as an integrating factor.

Thus, $\frac{d}{dx}(x^2 \cdot w) = -2x \Rightarrow x^2 w = -x^2 + c_1$

Substituting $w = u^{-2}$ we have $u = \frac{x}{\sqrt{c_1 - x^2}}$

So $y' = \frac{x}{\sqrt{c_1 - x^2}}$ and finally $y = \sqrt{c_1 - x^2} + c_2$.

1-114

Solve the following differential equations by suitable substitutions.

(a) $\frac{dy}{dx} + x^3 (y-x)^2 = 1$,

(b) $\frac{dy}{dx} = \frac{x}{x^2 y + y^3}$.

****

(a) Let $u = y - x$, then $\frac{dy}{dx} = \frac{du}{dx} + 1$. The differential equation becomes

$$\frac{du}{dx} + x^3 u^2 = 0$$

which is a separable equation. We write the equation

$$\frac{du}{u^2} + x^3 dx = 0$$

and integrate

$$\frac{1}{u} = \frac{x^4}{4} + c$$

where c is a constant of integration. Put back $u = y - x$

$$\frac{1}{y-x} = \frac{x^4}{4} + c$$

or

$$y = x + \frac{1}{\frac{x^4}{4} + c}.$$

(b) Let $u = x^2$, then $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = 2x\frac{dy}{du}$. The differential equation becomes

$$2\frac{dy}{du} = \frac{1}{uy + y^3}$$

If y is considered as the dependent variable, then the equation is non-linear since y^3 term appears. However if we change the role of u and y, that is,

$$w = -\frac{x^2}{2}e^{2x} + \frac{x}{4}e^{2x} + Cx$$

$$\frac{1}{y} = \frac{-2x^2e^{2x} + xe^{2x} + 4x}{4}$$

$$y = \frac{4}{-2x^2e^{2x} + xe^{2x} + C_1x}$$

1-113

Solve the following differential equation

$$\frac{dy}{dx} = \frac{x+y+1}{x+y-1}, \quad y(0)=1.$$

****

Let $u = x+y$ so that $\frac{dy}{dx} = \frac{du}{dx} - 1$. the differential equation becomes

$$\frac{du}{dx} - 1 = \frac{u+1}{u-1}$$

or $$\frac{du}{dx} = \frac{2u}{u-1}.$$

It is of separable type

$$\int \frac{u-1}{u}du = 2\int dx$$

which gives

$$u - \ln|u| = 2x + C.$$

Using the initial condition $y(0)=1$ i.e. $u(0)=1$, this gives $C=1$. The final solution

$$x+y-\ln|x+y| = 2x+1.$$

1-116

Solve the differential equation $\frac{dy}{dx} = \frac{x + y^2}{2y}$. (1)

* *

The equation can be written as

$$\frac{dy}{dx} - \frac{1}{2}y = \frac{x}{2y}. \qquad (2)$$

This is a Bernoulli equation of the form

$$\frac{dy}{dx} + p(x)y = q(x)y^n$$

which can be transformed into a linear equation by means of the substitution

$$u = y^{1-n}.$$

In the present case, with $n = -1$, we set

$$u = y^2 \quad \text{i.e.} \quad y = \sqrt{u}.$$

Then since

$$\frac{dy}{dx} = \frac{1}{2\sqrt{u}} \frac{du}{dx},$$

equation (2) reduces to the equation

$$\frac{1}{2\sqrt{u}} \frac{du}{dx} - \frac{1}{2}\sqrt{u} = \frac{x}{2\sqrt{u}}.$$

Multiplying both sides by $2\sqrt{u}$, we obtain the linear equation

$$\frac{du}{dx} - u = x. \qquad (3)$$

This equation has an integrating factor

we treat u as the dependent variable instead of y, then the differential equation can be rewritten as

$$\frac{du}{dy} = 2yu + 2y^3$$

which is a standard linear first order equation. The integrating factor is $e^{\int -2y\,dy} = e^{-y^2}$. On multiplying both sides by the integrating factor

$$\frac{d}{dy}(e^{-y^2}u) = 2y^3 e^{-y^2}$$

and on integrating

$$e^{-y^2}u = -e^{-y^2}(y^2+1) + C.$$

We write the final solution as

$$x^2 = u = -(y^2+1) + Ce^{y^2}.$$

1-115

Find a differential equation whose solutions have the form: $y = \sqrt{c+x}$. Will the functions "$y = -\sqrt{c+x}$" also give solutions of that differential equation? Why or why not?

****

To manipulate the equation so that c can be eliminated by differentiation, square it:

$$y^2 = c + x$$

Differentiating, $2yy' = 1$.

The functions "$y = -\sqrt{c+x}$" will also give solutions, since squaring still gives $y^2 = c+x$.

Equation (2a) now reduces to

$$\frac{du}{dx} - 1 = u$$

which is a separable equation. We may write

$$\frac{du}{u+1} = dx .$$

Integrating this last equation, we have

$$\ln (u+1) = x + c.$$

This can be simplified to give

$$x + y^2 + 1 = Ae^x$$

where $A = e^c$ is an arbitrary constant.

Therefore, we obtain the result

$$y = (Ae^x - x - 1)^{1/2},$$

as before.

1-117

Suppose $y = y(x)$ is a solution to the initial value problem

$$y' = x + y^2; \quad y(1) = 2.$$

What are $y'(1)$ and $y''(1)$?

****

$$y'(1) = 1 + y(1)^2 = 1 + 2^2 = \underline{\underline{5}}$$

e^{-x}. (see the section on linear equations.)

Therefore, if we multiply both sides of equation (3) by e^{-x}, we have

$$\frac{d}{dx}\left(e^{-x}u\right) = xe^{-x}.$$

Accordingly,

$$e^{-x}u = \int xe^{-x}\,dx + c.$$

Evaluating the integral on the right hand side using the method of integration by parts, we find after simplification

$$u = -(x+1) + ce^{x}$$

so that

$$y^2 = -(x+1) + ce^{x}$$

or

$$y = (ce^{x} - x - 1)^{1/2}.$$

Alternate Method

Write equation (1) in the following form:

$$2y\frac{dy}{dx} = x + y^2. \qquad (2a)$$

Make the substitution

$$x + y^2 = u$$

so that

$$1 + 2y\frac{dy}{dx} = \frac{du}{dx}. \qquad (3a)$$

1-119

Find the general solution of the equation

$$(x^2y^3+ y)dx = (x^3y^2 -x)dy \qquad (1)$$

****

Try to use some differential formulas to solve the equation, i.e. to make both side of (1) differentials of something. Rewrite the equation as

$$y\,dx + x\,dy = x^3y^2\,dy - x^2y^3\,dx \qquad (2)$$

The left-hand side is the differential of xy, but the right-hand side is not yet a differential. Try to divide (2) by some power of xy (This keeps the left-hand side a differential) to make the right-hand side a differential. If we divide (2) by $(xy)^2$, the right-hand side becomes

$$x\,dy - y\,dx$$

This is not a differential. Try again by dividing by $(xy)^3$, we get

$$\frac{d(xy)}{(xy)^3} = \frac{dy}{y} - \frac{dx}{x} \qquad (3)$$

Now both side of (3) are differentials, i.e. (3) is

$$d\left(\frac{-1}{2(xy)^2}\right) = d(\ln y - \ln x)$$

Integrate both side, we obtain the general solution of (1) as

$$\frac{1}{2(xy)^2} = \ln\frac{x}{y} + C$$

As for y'', differentiate the equation, to get

$$y'' = 1 + 2yy', \text{ so } y''(1) = 1 + 2y(1)y'(1),$$

$$y''(1) = 1 + 2 \cdot 2 \cdot 5 = \underline{\underline{21}}$$

1-118

Solve $x^2y' - xy = y^2 \ln x$.

****

Dividing by x^2 yields the Bernoulli type D.E. $y' - \frac{1}{x}y = \frac{y^2 \ln x}{x^2}$. Use the substitutions $w = y^{-1}$, $y' = -\frac{1}{w^2}w'$ to get the linear first order D.E:

$$w' + \frac{1}{x}w = -\frac{\ln x}{x^2}.$$

The integrating factor is x.

$$xw' + w = -\frac{\ln x}{x}$$

$$(xw)' = -\frac{\ln x}{x}.$$

Integration yields

$$xw = -\tfrac{1}{2}[\ln x]^2 + C$$

$w = 1/y$ yields

$$\frac{x}{y} = -\frac{1}{2}[\ln x]^2 + C$$

Solving for y

$$y = \frac{2x}{2C - [\ln x]^2}.$$

1-121

Solve: $y' = y - xy^3 e^{-2x}$.

****

WRITTEN $y' - y = -xe^{-2x} y^3$, THIS IS RECOGNIZABLE AS A BERNOULLI DIFFERENTIAL EQUATION. IN A DIFFERENTIAL EQUATION OF THE FORM $y' + P(x)y = Q(x)y^n$ (THE "BERNOULLI" FORM), A CHANGE OF DEPENDENT VARIABLE $z = y^{1-n}$ WILL LEAD TO THE LINEAR EQUATION $z' + (1-n)zP(x) = (1-n)Q(x)$ WHICH IS READILY SOLVABLE.

IN THE GIVEN CASE, $n = 3$, $z = y^{1-3} = y^{-2}$ AND WE OBTAIN $z' + 2z = 2xe^{-2x}$. HERE, $P(x) = 2$ AND $Q(x) = 2xe^{-2x}$ AND

$$z = e^{-\int P(x)dx} \int e^{\int P(x)dx} Q(x)\,dx + Ce^{-\int P(x)dx}.$$

$$\int P(x)dx = \int 2\,dx = 2x, \quad e^{\int P(x)dx} = e^{2x}, \quad e^{-\int P(x)dx} = e^{-2x}.$$

$$\int e^{\int P(x)dx} Q(x)\,dx = \int e^{2x} \cdot 2xe^{-2x}\,dx = \int 2x\,dx = x^2.$$

$$\therefore z = e^{-2x}(x^2) + Ce^{-2x}. \quad \text{SINCE } z = y^{-2}, \quad y^{-2} = x^2 e^{-2x} + Ce^{-2x}$$

OR $e^{2x} = x^2 y^2 + Cy^2$.

1-120

(a) For the differential equation $y' = y(1 - y)$, construct the direction field.

(b) Sketch the solution (or integral) curves through

(i) $(1, \frac{3}{2})$

(ii) $(1, \frac{1}{2})$, and

(iii) $(1, -1)$.

****

$y' = y(1-y) = 0 \Rightarrow y = 0$ or $y = 1$.

If $y = 2$, $y' = 2(1-2) = -2$. at any point on the line $y = 2$. Hence, any solution crossing $y = 2$ would have slope equal to -2 there.

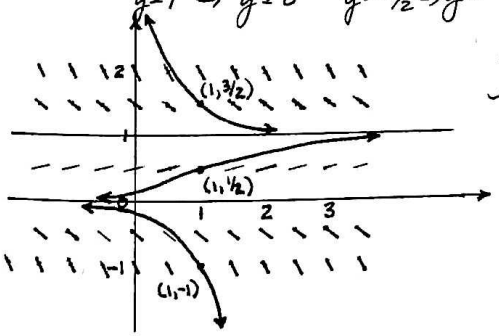
If $y = \frac{3}{2} \Rightarrow y' = \frac{3}{2}(1 - \frac{3}{2}) = -\frac{3}{4}$. Note as $y \to 1^+ \Rightarrow y' \to 0^-$

Hence, solutions must become asymptotic to the line $y = 1$ (from above). Since as $y \to 1^- \Rightarrow y' \to 0^+$, the solutions also become asymptotic from below 1. Similarly, as $y \to 0^+ \Rightarrow y' \to 0^+$, solutions become asymptotic to $y = 0$ from above and from below (as $y \to 0^- \Rightarrow y' \to 0^-$).

Data: If $y = 2 \Rightarrow y' = -2$ $\quad y = \frac{1}{2} \Rightarrow y' = \frac{1}{4}$ $\quad y = -1 \Rightarrow y' = -2$

$y = \frac{3}{2} \Rightarrow y' = -\frac{3}{4}$ $\quad y = 0 \Rightarrow y' = 0$

$y = 1 \to y' = 0$ $\quad y = -\frac{1}{2} \Rightarrow y' = -\frac{3}{4}$



In general, the solutions 'flow' with the direction fields.

1-123

Solve $\frac{dy}{dx} = \frac{4y - 3x - 2}{2x - y + 3}$.

****

Set $\begin{cases} x = \xi + h \\ y = \eta + k \end{cases}$, $\frac{d\eta}{d\xi} = \frac{4\eta - 3\xi + 4k - 3h - 2}{2\xi - \eta - k + 2h + 3}$

We want

$$\begin{cases} 4k - 3h - 2 = 0 \\ -k + 2h + 3 = 0 \end{cases}$$

$k = -1$, $h = -2$, and

$\frac{d\eta}{d\xi} = \frac{4\eta - 3\xi}{2\xi - \eta}$, set $\frac{\eta}{\xi} = v$, we have

$$v + \xi \frac{dv}{d\xi} = \frac{4v - 3}{2 - v}$$

$$\xi \frac{dv}{d\xi} = \frac{4v - 3 - 2v + v^2}{2 - v} = \frac{(v+3)(v-1)}{2-v}$$

$$\left(\frac{1}{v-1} - \frac{5}{v+3} \right) dv = \frac{4\, d\xi}{\xi}$$

$$\ln \left| \frac{v-1}{(v+3)^5} \right| = \ln |C \xi^4|$$

by $v = \frac{\eta}{\xi}$, $|\eta - \xi| = C\, |\eta + 3\xi|^5$

by $\xi = x + 2$, $\eta = y + 1$, we have

$$|y - x - 1| = C\, |y + 3x + 7|^5.$$

1-122

Find the general solution: $x\frac{dy}{dx} - 3y - xy^{1/2} = 0$

****

$$x\frac{dy}{dx} - 3y = xy^{1/2}$$

$$xy^{-1/2}\frac{dy}{dx} - 3y^{1/2} = x$$

$$y^{-1/2}\frac{dy}{dx} - \frac{3}{x}y^{1/2} = 1$$

THIS IS A BERNOULLI EQUATION WITH $v = y^{1/2}$, $\frac{dv}{dx} = \frac{1}{2}y^{-1/2}\frac{dy}{dx}$.
MULTIPLY BY $\frac{1}{2}$ AND SUBSTITUTE TO CREATE A LINEAR EQUATION IN x AND v:

$$\frac{1}{2}y^{-1/2}\frac{dy}{dx} - \frac{3}{2x}y^{1/2} = \frac{1}{2}$$

$$\frac{dv}{dx} - \frac{3}{2x}v = \frac{1}{2}$$

(INTEGRATING FACTOR: $\mu(x) = e^{\int \frac{-3}{2x}dx} = e^{-\frac{3}{2}\ln x} = x^{-3/2}$)

$$x^{-3/2}\frac{dv}{dx} - \frac{3}{2}x^{-5/2}v = \frac{1}{2}x^{-3/2}$$

$$\frac{d}{dx}\left(x^{-3/2}v\right) = \frac{1}{2}x^{-3/2}$$

$$x^{-3/2}v = \int \frac{1}{2}x^{-3/2}dx = -x^{-1/2} + C$$

$$v = x^{3/2}\left(C - x^{-1/2}\right)$$

$$= Cx^{3/2} - x$$

REPLACING v BY $y^{1/2}$, WE OBTAIN THE GENERAL SOLUTION:

$$y^{1/2} = Cx^{3/2} - x$$

OR $$y = \left(Cx^{3/2} - x\right)^2.$$

1-125

Find the general solution: $(x - y - 4)dx + (x - y)dy = 0.$

****

SINCE THE EXPRESSION $X-Y$ APPEARS IN BOTH TERMS, WE MAKE THE SUBSTITUTION $z = x - y$.
DIFFERENTIATING, WE GET $dz = dx - dy$, OR $dy = dx - dz$.

THIS LEADS TO A SEPARABLE EQUATION IN X AND z WHICH WE SOLVE AS FOLLOWS:

$$(z-4)dx + z(dx-dz) = 0$$
$$(z-4)dx + z\,dx - z\,dz = 0$$
$$(2z-4)dx - z\,dz = 0$$
$$2(z-2)dx - z\,dz = 0$$
$$2\,dx - \frac{z}{z-2}\,dz = 0$$

WE DIVIDE TO OBTAIN $\frac{z}{z-2} = 1 + \frac{2}{z-2}$, THEN INTEGRATE.

$$\int 2\,dx - \int \left[1 + \frac{2}{z-2}\right] dz = \int 0$$
$$2x - z - 2\ln|z-2| = C$$

NOW SUBSTITUTING FOR z, WE OBTAIN THE SOLUTION IN TERMS OF X AND Y:

$$2x - (x-y) - 2\ln|(x-y)-2| = C$$
$$x + y - 2\ln|x-y-2| = C$$

1-124

(a) Find the family of general solutions of the Clairaut equations $y = px + \frac{1}{2}p^2$ where $p = \frac{dy}{dx}$.

(b) Find a particular solution y_p, which is not a member of the above family.

(c) If y_c is a member of the family in (a), then show that y_c is tangent to y_p.

****

Differentiate $y = px + \frac{1}{2}p^2$ with respect to x

$$\frac{dy}{dx} = \frac{dp}{dx}x + p \cdot 1 + p \cdot \frac{dp}{dx}$$

This gives us $(x+p)\frac{dp}{dx} = 0$, because $\frac{dy}{dx} = p$.

(a) If $\frac{dp}{dx} = 0$, then $p = c$ where c is an arbitrary constant. Substituting $p = c$ in the original equation, we obtain the family of general solutions

$$y_c = cx + \frac{1}{2}c^2, \quad c \text{ arbitrary.}$$

(b) If $x + p = 0$, then $p = -x$. Substituting $p = -x$ in the original equation, we obtain the particular solution

$$y_p = -x^2 + \frac{1}{2}x^2 = -\frac{1}{2}x^2.$$

(c) Solve $y_c = y_p$, i.e. $cx + \frac{1}{2}c^2 = -\frac{1}{2}x^2$

This gives us $(x+c)^2 = 0$, i.e. $x = -c$ (twice) and y_c and y_p have exactly one common point, $(-c, -\frac{1}{2}c^2)$. Further,

$$\frac{dy_c}{dx} = c \quad \text{and} \quad \frac{dy_p}{dx} = -x = c \quad \text{at } x = -c.$$

Thus y_c is tangent to y_p at the point $(-c, -\frac{1}{2}c^2)$.

2-2

Solve the initial-value problem:

$$\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 53y = 0 \quad \text{with } y(0)=3,\ y'(0)=0.$$

****

THE CHARACTERISTIC EQUATION $m^2 + 4m + 53 = 0$ HAS ROOTS

$$m = \frac{-4 \pm \sqrt{16 - 4\cdot 53}}{2} = \frac{-4 \pm \sqrt{-196}}{2} = -2 \pm 7i$$

SO THE GENERAL SOLUTION IS

$$y = (c_1 \cos 7x + c_2 \sin 7x)\, e^{-2x}$$

$$y' = \left[(-2c_1 + 7c_2)\cos 7x + (-2c_2 - 7c_1)\sin 7x\right] e^{-2x}$$

EVALUATING EACH AT $X=0$, WE OBTAIN THE EQUATIONS

$$3 = y(0) = c_1$$
$$0 = y'(0) = -2c_1 + 7c_2$$

THUS $c_1 = 3$ AND $c_2 = \frac{2}{7}c_1 = \frac{2}{7}(3) = \frac{6}{7}$

SPECIFIC SOLUTION:

$$y = \left(3\cos 7x + \frac{6}{7}\sin 7x\right) e^{-2x}$$

2-3

Note that $y_1 = \sin(3x)$ and $y_2 = \sin(3x + \pi)$ are two different solutions of the differential equation $y'' + 9y = 0$. Determine whether or not $y = C_1y_1(x) + C_2y_2(x)$ is the general solution of the differential equation. Justify your answer.

****

2
SECOND ORDER LINEAR EQUATIONS

CONSTANT COEFFICIENT, HOMOGENEOUS

2-1

Solve the following initial value problem. y" + 6y' + 9y = 0, y(0)=0, y'(0)=3.

****

characteristic equation: $r^2+6r+9=0$

$(r+3)^2=0$

$r=-3$, double root

general homogeneous solution: $y = c_1 e^{-3x} + c_2 x e^{-3x}$

$y(0)=0$: $0 = c_1 + 0$ $\quad c_1 = 0$

so $y = c_2 x e^{-3x}$

$y'(0)=3$: $y' = c_2 e^{-3x} - 3c_2 x e^{-3x}$

$3 = c_2 - 0$ $\quad c_2 = 3$

initial value problem solution: $y = 3x e^{-3x}$

The characteristic equation is $r^2+2r+1=(r+1)^2=0$, so $u = c_1 e^{-x} + c_2 x e^{-x}$. Since $u(0) = c_1 = 1$, $u = e^{-x} + c_2 x e^{-x} = (1 + c_2 x) e^{-x}$. This goes negative for large x when $c_2 < 0$. Now $u' = -e^{-x} + c_2 e^{-x} - c_2 x e^{-x}$, so $u'(0) = -1 + c_2$, and $c_2 = u'(0) + 1$. The condition is $u'(0) + 1 < 0$, or $u'(0) < -1$. $\underline{\underline{V_0 = -1.}}$

2-5

Solve the differential equation $y'' - 5y' + 6y = 0$

****

The equation is a linear homogeneous equation of second order with constant coefficients.

Assume $y = e^{rt}$ is a solution of the given equation.

Then substituting $y = e^{rt}$, $y' = r e^{rt}$ and $y'' = r^2 e^{rt}$ in the equation, we obtain

$$e^{rt}(r^2 - 5r + 6) = 0.$$

that is $r^2 - 5r + 6 = 0$, the characteristic equation

$$(r-2)(r-3) = 0$$

Therefore $r = 2, 3$

The roots of the characteristic equation are real and distinct.

The two solutions of the given diff. eq. are e^{2t} and e^{3t} which are linearly independent.

Therefore the general solution of the given homogeneous equation is

$$Y(t) = C_1 e^{2t} + C_2 e^{3t},$$ where C_1, C_2 are two arbitrary constants.

Since y_1 and y_2 are solutions of the D.E., it suffices to determine whether the Wronskian is zero or nonzero.

$$W(y_1, y_2) = \begin{vmatrix} \sin 3x & \sin(3x+\pi) \\ 3\cos 3x & 3\cos(3x+\pi) \end{vmatrix}$$

$$= 3(\sin 3x \cos(3x+\pi) - \cos 3x \sin(3x+\pi))$$
$$= 3\sin(3x - (3x+\pi)) = 3\sin(-\pi) = 0.$$

y_1 and y_2 do not form a fundamental set of solutions.

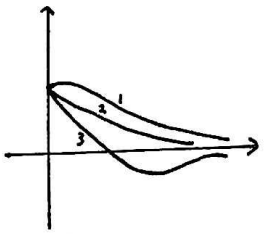
(Comment: if $y_3(x) = \sin(3x + \frac{\pi}{2})$, then y_1 and y_3 do form a fundamental set of solutions.)

2-4

Suppose u is a solution to the damped-vibration equation:

$$u'' + 2u' + u = 0; \quad u(0) = 1, \ \ldots \text{ (critical damping)}$$

There is a v_0 so that if $u'(0) \geq v_0$, then u(t) stays positive for all t, but if $u'(0) < v_0$, then u "overshoots" and becomes negative for sufficiently large t (e.g. curve 3). Find v_0.



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2-7

For the following differential equations, find the general solutions.

(a) $y'' + 3y' + 2y = 0$

(b) $y'' + 2y' + 2y = 0$

****

(a) The characteristic (or auxiliary) equation is:

$r^2+3r+2=0$

$(r+2)(r+1)=0 \Rightarrow r=-2,\ r=-1$

$\therefore e^{-2t}, e^{-t}$ are 2 linearly independent solutions.

$\therefore y = c_1 e^{-2t} + c_2 e^{-t}$ is the general solution.

(b) The characteristic equation is:

$r^2+2r+2=0$

$$r = \frac{-2 \pm \sqrt{4-4(1)(2)}}{2} = \frac{-2 \pm \sqrt{-4}}{2} = \frac{-2 \pm 2i}{2} = -1 \pm i$$

$\therefore e^{(-1+i)t}, e^{(-1-i)t}$ are solutions. Usually this is not the best form to use. Both the real and imaginary parts of these functions are also solutions:

$$\therefore e^{(-1+i)t} = e^{-t+it} = e^{-t}e^{it} = e^{-t}\left[\cos t + i\sin t\right] \quad (e^{i\beta} = \cos\beta + i\sin\beta)$$

$$= e^{-t}\cos t + \left(e^{-t}\sin t\right)i$$

$\therefore e^{-t}\cos t$ and $e^{-t}\sin t$ are 2 real linearly independent solutions. Hence, the general solution (in real form) can be written $y = c_1 e^{-t}\cos t + c_2 e^{-t}\sin t$

2-8

Solve the following differential equations:

a) $y'' + 4y' = 0.$

b) $y'' + 4y' + 4y = 0.$

c) $y'' + 4y' - 5y = 0.$

d) $y'' + 4y' + 5y = 0.$

****

2-6

Solve the following initial value problem: $y'' + 4y' + 5y = 0$; $y(0) = 2$, $y'(0) = -5$.

****

This is a 2nd order linear constant coefficient DE. We first solve the auxiliary quadratic eq. $r^2 + 4r + 5 = 0$, $r = \frac{-4 \pm \sqrt{(4)^2 - 4(1)(5)}}{2(1)}$

$r = \frac{-4 \pm \sqrt{-4}}{2} = \frac{-4 \pm 2i}{2} = -2 \pm i$

where $i = \sqrt{-1}$.

The general solution to the DE for complex roots $r = \alpha \pm \beta i$ is $y = e^{\alpha x}(C_1 \cos\beta x + C_2 \sin\beta x)$.

So our general solution is

$y = e^{-2x}(C_1 \cos x + C_2 \sin x)$.

To solve for C_1 and C_2 we compute y'.

$y' = e^{-2x}(-C_1 \sin x + C_2 \cos x) - 2e^{-2x}(C_1 \cos x + C_2 \sin x)$

$= e^{-2x}\left[(-C_1 - 2C_2)\sin x + (C_2 - 2C_1)\cos x\right]$

From the initial conditions we have

$2 = y(0) = e^{-2(0)}(C_1 \cos(0) + C_2 \sin(0)) = C_1$

$-5 = y'(0) = e^{-2(0)}\left[(-C_1 - 2C_2)\sin(0) + (C_2 - 2C_1)\cos(0)\right] = C_2 - 2C_1$

So $C_1 = 2$, $C_2 - 2(2) = -5$ and $C_2 = -1$ and

$y = e^{-2x}(2\cos x - \sin x)$.

THE ROOTS $m_1 = 0$, $m_2 = 2$, AND $m_3 = -1$ DETERMINE THE GENERAL SOLUTION, SO

$$y = c_1 + c_2 e^{2x} + c_3 e^{-x}$$
$$y' = 2c_2 e^{2x} - c_3 e^{-x}$$
$$y'' = 4c_2 e^{2x} + c_3 e^{-x}$$

EVALUATING AT $x = 0$ AND SOLVING FOR c_1, c_2, AND c_3, WE OBTAIN THE SPECIFIC SOLUTION.

$$\left.\begin{aligned} 0 = y(0) &= c_1 + c_2 + c_3 \\ 6 = y'(0) &= 2c_2 - c_3 \\ 0 = y''(0) &= 4c_2 + c_3 \end{aligned}\right\} \longrightarrow 6c_2 = 6$$

$$\Longrightarrow \begin{cases} c_2 = 1 \\ c_3 = -4 \\ c_1 = 3 \end{cases}$$

$$\therefore \quad y = 3 + e^{2x} - 4e^{-x}$$

2-10

Solve the differential equation $y'' - 6y' + 9y = 0$

****

The equation being a linear homogeneous equation of second order with constant coefficients, we assume a solution $y = e^{rt}$

The characteristic equation is

$$r^2 - 6r + 9 = 0$$
$$(r-3)^2 = 0$$

Therefore $r = 3$ is a double root of the characteristic equation.

Therefore the general solution of the given

a) the auxiliary equation will be

$m^2 + 4m = 0$

$m(m+4) = 0$

$m = 0, -4$ so $y = c_1 e^0 + c_2 e^{-4x} = c_1 + c_2 e^{-4x}$

b) $m^2 + 4m + 4 = 0$

$(m+2)(m+2) = 0$

$m = -2, -2$ so $y = c_1 e^{-2x} + c_2 x e^{-2x}$

c) $m^2 + 4m - 5 = 0$

$(m+5)(m-1) = 0$

$m = -5, 1$ so $y = c_1 e^{-5x} + c_2 e^{x}$

d) $m^2 + 4m + 5 = 0$

$$m = \frac{-4 \pm \sqrt{4^2 - 4 \cdot 1 \cdot 5}}{2 \cdot 1}$$

$$= \frac{-4 \pm \sqrt{16-20}}{2} = \frac{-4 \pm \sqrt{-4}}{2}$$

$$= \frac{-4 \pm 2i}{2} = -2 \pm i$$

so $y = e^{-2x}(c_1 \sin x + c_2 \cos x)$

2-9

Solve the initial-value problem:

$$\frac{d^3y}{dx^3} - \frac{d^2y}{dx^2} - 2\frac{dy}{dx} = 0 \qquad \text{with } y(0)=0,\ y'(0)=6, \text{ and } y''(0)=0.$$

****

THE CHARACTERISTIC EQUATION FACTORS EASILY AS FOLLOWS:

$$m^3 - m^2 - 2m = m(m^2 - m - 2) = m(m-2)(m+1) = 0$$

2-12

Solve the initial value problem

$$\begin{cases} u'' + 2u' + 2u = 0 \\ u(0) = 2,\ u'(0) = 0 \end{cases}$$

and write down the solution in the form $u(t) = Re^{\alpha t}\cos(\mu t - \delta)$ and draw a graph.

****

$r^2+2r+2=0$, $\quad r=\dfrac{-2\pm\sqrt{4-8}}{2} = -1 \pm i$

the general solution is

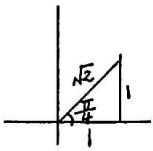
$$u(t) = e^{-t}(A\cos t + B\sin t)$$

$$u'(t) = e^{-t}\left[(B-A)\cos t - (A+B)\sin t\right]$$

by using the initial value condition we have

$$u(0) = A = 2$$

$$u'(0) = B - A = 0, \quad B = A = 2$$



Thus

$$u(t) = 2e^{-t}(\cos t + \sin t)$$

$$= 2\sqrt{2}\, e^{-t}\left(\frac{1}{\sqrt{2}}\cos t + \frac{1}{\sqrt{2}}\sin t\right)$$

$$= 2\sqrt{2}\, e^{-t}\left(\cos t\cos\frac{\pi}{4} + \sin t\sin\frac{\pi}{4}\right)$$

$$= 2\sqrt{2}\, e^{-t}\cos\left(t - \frac{\pi}{4}\right)$$

i.e. $\quad R = 2\sqrt{2}$, $\alpha = -1$, $\mu = 1$, $\delta = \dfrac{\pi}{4}$.

equation is

$$y(t) = c_1 e^{3t} + c_2 t e^{3t},$$

where e^{3t} and te^{3t} are the two linearly independent solutions of the given diff. equation and c_1, c_2 are two arbitrary constants.

2-11

Find a general solution of the differential equation

$$y'' + 4y' + 13y = 0.$$

* *

Assuming a solution of the form $y = e^{mx}$ and substituting it into the differential equation, we obtain

$$(m^2 + 4m + 13) e^{mx} = 0.$$

Thus $y = e^{mx}$ will be a solution if m satisfies the equation

$$m^2 + 4m + 13 = 0.$$

Using the quadratic formula, we find

$$m = \frac{1}{2}(-4 \pm \sqrt{-36})$$

$$= -2 \pm 3i.$$

Therefore the general solution is given by

$$y(x) = e^{-2x}(c_1 \cos 3x + c_2 \sin 3x),$$

where c_1 and c_2 are arbitrary constants.

2-14

Solve the initial value problem

$$y'' + \sqrt{2}y' - 4y = 0.$$

$$y(0) = 0,\ y'(0) = 1.$$

****

Since the equation has constant coefficients,

Set $y = e^{rx} \Rightarrow r^2 + \sqrt{2}r - 4 = 0.$

Using the quadratic formula

$$r = \frac{-\sqrt{2} \pm \sqrt{(\sqrt{2})^2 - 4(-4)}}{2} = \frac{-\sqrt{2} \pm \sqrt{2+16}}{2}$$

$$r = -\frac{\sqrt{2} \pm 3\sqrt{2}}{2} = \sqrt{2}, -2\sqrt{2}$$

General solution $y = c_1 e^{\sqrt{2}x} + c_2 e^{-2\sqrt{2}x}$

Initial conditions $y(0) = c_1 + c_2 = 0 \Rightarrow \sqrt{2}c_1 + \sqrt{2}c_2 = 0$

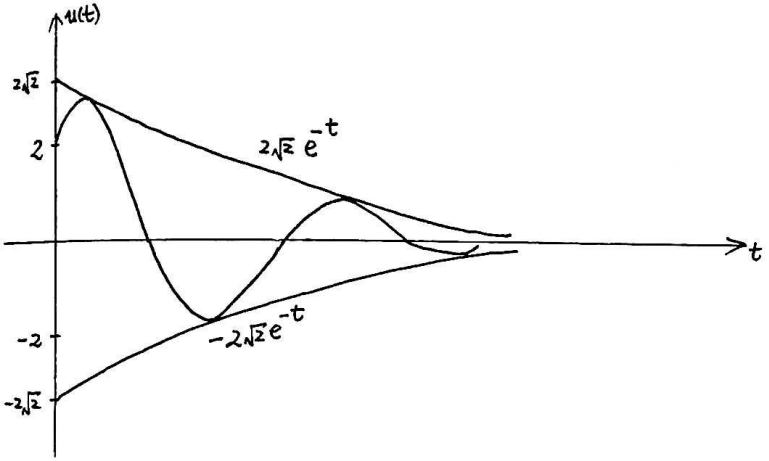
$y'(0) = \sqrt{2}c_1 - 2\sqrt{2}c_2 = 1 \Rightarrow \sqrt{2}c_1 - 2\sqrt{2}c_2 = 1$

Subtract the equations to get $3\sqrt{2}c_2 = -1$

$\Rightarrow c_2 = -\frac{1}{3\sqrt{2}}$ and since $c_1 + c_2 = 0$,

$$c_1 = \frac{1}{3\sqrt{2}}$$

$$\boxed{y = \frac{1}{3\sqrt{2}}\left(e^{\sqrt{2}x} - e^{-2\sqrt{2}x}\right)}$$



2-13

Find the general solution of the equation

$$y'' + 2y' + y = 0$$

****

This is an equation of constant coefficients. The corresponding auxiliary equation is

$$m^2 + 2m + 1 = 0$$

It has two equal real roots $m=-1$. Hence the general solution is

$$C_1 e^{-x} + C_2 x e^{-x}.$$

2-16

Given the differential equation $y'' - 4y = 0$ and the corresponding conditions $x = 0$ when $y = -1$ and $y \to 0$ as $x \to \infty$, what is y when $x = \ln 3$?

****

The auxiliary equation may be written as

$$m^2 - 4 = 0 \quad \text{or} \quad m = \pm 2$$

Therefore the general solution of the differential is

$$y = c_1 e^{-2x} + c_2 e^{2x}$$

The condition $x = 0$, when $y = -1$ yields

1) $\quad -1 = c_1 + c_2$

The condition $y \to 0$ as $x \to \infty$ yields

2) $\quad 0 = \lim_{x \to \infty} c_1 e^{-2x} + \lim_{x \to \infty} c_2 e^{2x}$

Condition 2) dictates that $c_2 = 0$

condition 1) yields $c_1 = -1$

Therefore the particular solution satisfying the given conditions is

$$y = -e^{-2x}$$

@ $x = \ln 3 \quad y = -e^{-2\ln 3}$

$$y = -\frac{1}{9}$$

2-15

Obtain a general solution to $y^{(4)} - 7y''' + 11y'' + 5y' - 14y = 0$.

****

WE HAVE $(D^4 - 7D^3 + 11D^2 + 5D - 14)y = 0$ IF $D = \frac{d}{dx}$.

$\therefore\ m^4 - 7m^3 + 11m^2 + 5m - 14 = 0$ IS THE AUXILIARY EQUATION, THE ROOTS OF WHICH LEAD TO A GENERAL SOLUTION OF THE GIVEN EQUATION. POSSIBLE RATIONAL ROOTS ARE $\pm 1, \pm 2, \pm 7, \pm 14$ (FROM "ZEROES OF POLYNOMIAL FUNCTIONS" IN ALGEBRA). SHOWING ONLY THE SUCCESSFUL SYNTHETIC DIVISIONS,

2	1	−7	11	5	−14
		2	−10	2	14
	1	−5	1	7	0

−1	1	−5	1	7
		−1	6	−7
	1	−6	7	0

WE HAVE $m = 2$, $m = -1$, AND $m^2 - 6m + 7 = 0$. USING THE QUADRATIC FORMULA ON THE LATTER, $m = 3 \pm \sqrt{2}$.

THEN e^{2x}, e^{-x}, $e^{(3+\sqrt{2})x}$, $e^{(3-\sqrt{2})x}$ ARE SOLUTIONS OF THE GIVEN EQUATION. THERE ARE FOUR; IT IS A 4TH ORDER DIFFERENTIAL EQUATION; SO IF THEY ARE LINEARLY INDEPENDENT, A LINEAR COMBINATION OF ALL FOUR IS A GENERAL SOLUTION. THE WRONSKIAN DETERMINANT

$$W\left[e^{2x}, e^{-x}, e^{(3+\sqrt{2})x}, e^{(3-\sqrt{2})x}\right] = -84\sqrt{2}\,e^{7x}$$ DOES NOT VANISH

SO THE SOLUTIONS ARE LINEARLY INDEPENDENT AND

$$y = C_1 e^{2x} + C_2 e^{-x} + C_3 e^{(3+\sqrt{2})x} + C_4 e^{(3-\sqrt{2})x}$$ IS THE GENERAL SOLUTION. IF WE SET $C_3 = \frac{K_1 + K_2}{2}$ AND $C_4 = \frac{K_1 - K_2}{2}$,

$$y = C_1 e^{2x} + C_2 e^{-x} + K_1 e^{3x}\cosh\sqrt{2}x + K_2 e^{3x}\sinh\sqrt{2}x$$

WHERE K_1 AND K_2 ARE ARBITRARY.

2-18

Solve $y'' + 3y' + 4y = 0$ with initial conditions $y(0) = 2$ and $y'(0) = -1$.

****

The auxiliary quadratic equation is $r^2 + 3r + 4 = 0$, which has <u>complex</u> roots $r_{1,2} = -\frac{3}{2} \pm \frac{\sqrt{7}}{2}i$. Hence the <u>real</u> general solution is $y = e^{-\frac{3}{2}x}\left[c_1 \cos\frac{\sqrt{7}}{2}x + c_2 \sin\frac{\sqrt{7}}{2}x\right]$.

Substituting $x = 0$ and $y = 2$ gives $c_1 = 2$. Before using the other initial condition, we must differentiate the general solution,

$$y' = e^{-\frac{3}{2}x}\left[-\frac{\sqrt{7}}{2}c_1 \sin\frac{\sqrt{7}}{2}x + \frac{\sqrt{7}}{2}c_2 \cos\frac{\sqrt{7}}{2}x\right] - \frac{3}{2}e^{-\frac{3}{2}x}\left[c_1 \cos\frac{\sqrt{7}}{2}x + c_2 \sin\frac{\sqrt{7}}{2}x\right].$$

Now substituting $x = 0$, $y' = -1$ and $c_1 = 2$, we get $-1 = \frac{\sqrt{7}}{2}c_2 - 3$, so $c_2 = 4/\sqrt{7}$.

Thus, the solution satisfying the given conditions is $y = e^{-\frac{3}{2}x}\left[2\cos\frac{\sqrt{7}}{2}x + \frac{4}{\sqrt{7}}\sin\frac{\sqrt{7}}{2}x\right]$.

2-19

Find a linear homogeneous differential equation with real constant coefficients for which a particular solution is $y = 31e^{3x}\sin 2x$

****

The roots of the auxiliary equation are $m = 3 \pm 2i$

2-17

Find general solutions for (a) $y^{(4)} + 8y'' + 16y = 0$.

(b) $y^{(4)} + 64y = 0$.

(c) $y^{(4)} + 4y''' + 8y'' = 0$.

****

(a) $(D^4+8D^2+16)y = 0 \qquad (D^4+8D^2+16) = (D^2+4)^2$.

A FACTOR OF THE FORM $[(D-A)^2+B^2]^n$, n A POSITIVE INTEGER, CONTRIBUTES ELEMENTS TO A SOLUTION SET OF THE FORMS $e^{Ax}\sin Bx$, $xe^{Ax}\sin Bx$, $x^2e^{Ax}\sin Bx$, $\cdots$ $x^{n-1}e^{Ax}\sin Bx$ AND $e^{Ax}\cos Bx$, $xe^{Ax}\cos Bx$, $\cdots$ $x^{n-1}e^{Ax}\cos Bx$. IF A AND B ARE NOT ZERO, THESE $2n$ FUNCTIONS ARE LINEARLY INDEPENDENT. IF $A=0$, $e^{Ax}=1$, AND THE $2n$ FUNCTIONS REMAIN INDEPENDENT.

$\therefore y = (C_1+C_2x)\cos 2x + (C_3+C_4x)\sin 2x$.

(b) $(D^4+64)y = 0. \qquad D^4+64 = D^4+16D^2+64-16D^2$

$$D^4+16D^2+64-16D^2 = (D^2+8)^2-(4D)^2 = (D^2+8-4D)(D^2+8+4D)$$

$$= [(D^2-4D+4)+4][(D^2+4D+4)+4]$$

$$= [(D-2)^2+2^2][(D+2)^2+2^2].$$

THEN FROM (a) ABOVE, $y = e^{2x}(C_1\cos 2x + C_2\sin 2x) + e^{-2x}(C_3\cos 2x + C_4\sin 2x)$.

(c) $(D^4+4D^3+8D^2)y = 0.$

$$D^4+4D^3+8D^2 = D^2(D^2+4D+8) = D^2[(D+2)^2+2^2].$$

SOLUTIONS CORRESPONDING TO D^n ARE $1, x, x^2, \cdots x^{n-1}$.

$\therefore y = C_1 + C_2x + C_3e^{-2x}\sin 2x + C_4e^{-2x}\cos 2x.$

2-21

Solve $y'' - 2my' + m^2y = 0$, given that m is a real constant.

****

The characteristic equation is $r^2 - 2mr + m^2 = 0$. This factors as $(r-m)^2 = 0$. Thus $r = m$ of multiplicity two. Hence

$$y = c_1 e^{mx} + c_2 x e^{mx}.$$

EULER EQUATIONS, HOMOGENEOUS

2-22

Find the general solution of $x^2y''+5xy'+8y=0$.

****

This is an Euler equation, with characteristic equation $r^2 + (5-1)r + 8 = r^2 + 4r + 8 = 0$. Solutions are $r = \frac{-4 \pm \sqrt{16-32}}{2} = -2 \pm 2j$, complex roots.

The corresponding general solution is

$$y(x) = |x|^{-2} \left[c_1 \cos(2 \ln|x|) + c_2 \sin(2 \ln|x|) \right]$$

$$= \frac{1}{x^2} \left[c_1 \cos(\ln x^2) + c_2 \sin(\ln x^2) \right]$$

The auxiliary equation is $(m-3-2i)(m-3+2i)=0$

$$(m-3)^2-4i^2=0$$

$$m^2-6m+9+4=0$$

$$m^2-6m+13=0$$

The differential equation is

$$\frac{d^2y}{dx^2}-6\frac{dy}{dx}+13y=0$$

2-20

Solve $y''-2y'+10y=0$

****

This is a homogeneous equation with constant coefficients.
We assume a solution $y(t)=e^{rt}$
The characteristic equation is

$$r^2-2r+10=0$$

Therefore $r=\dfrac{2\pm\sqrt{4-4(1)(10)}}{2}$

$$=\frac{2\pm i6}{2}=1\pm 3i$$

The roots of the characteristic eq. are the complex conjugates $1+3i$ and $1-3i$.
Therefore $e^t\cos 3t$ and $e^t\sin 3t$ are two linearly independent solutions of the given diff. equation.
The general solution is

$$y(t)=e^t(C_1\cos 3t+C_2\sin 3t)$$

2-24

Solve the following homogeneous, linear, second order ordinary differential equation possessing variable coefficients by each of the two methods described below.

$3x \cdot y'' + 2 \cdot y' = 0$; $y = y(x)$.

I Letting $y' \equiv \frac{dy}{dx} = Z$, reducing the given equation to a pair of coupled and separable first order differential equations.

II Multiplying the given equation by the factor x , thereby converting the equation into a Cauchy-Euler type of equation. Recall then, that the change of independent variable, letting $x = e^t$, converts the Cauchy-Euler equation into an equation with constant coefficients.

****

I Let $y' = z$. Then $y'' = z'$, and the given equation reduces to $\frac{dz}{dx} + \frac{2}{3} \cdot \frac{z}{x} = 0$.

Separation of Variables yields $3 \cdot \int \frac{dz}{z} + 2 \cdot \int \frac{dx}{x} = 0$,

and hence the solution $3 \cdot \ln z + 2 \cdot \ln x = \ln \hat{c}_1$,

or via logarithms, $z^3 \cdot x^2 = \hat{c}_1$.

Thus $z^3 \equiv (y')^3 = \frac{\hat{c}_1}{x^2}$, and $y' \equiv \frac{dy}{dx} = \left(\frac{\hat{c}_1}{x^2}\right)^{\frac{1}{3}}$.

This latter equation is also separable, giving

$\int dy = (\hat{c}_1)^{\frac{1}{3}} \cdot \int \frac{dx}{x^{2/3}}$, or

$y = \frac{(\hat{c}_1)^{\frac{1}{3}} \cdot x^{\frac{1}{3}}}{1/3} + c_2 = c_1 \cdot x^{\frac{1}{3}} + c_2$; $c_1 \equiv 3 \cdot (\hat{c}_1)^{\frac{1}{3}}$.

II The associated Cauchy-Euler differential equation is

$3x^2 \cdot y'' + 2x \cdot y' = 0$; $y = y(x)$.

2-23

Solve $x^2y'' - 3xy' + 4y = 0$

****

This is Euler's equation in which the coefficients are functions of x. We change it into an equation with constant coefficients by the substitution

$$x = e^z \text{ , that is } z = \ln x \text{ , } \frac{dz}{dx} = \frac{1}{x}$$

Then $$\frac{dy}{dx} = \frac{dy}{dz}\frac{dz}{dx} = \frac{1}{x}\frac{dy}{dz}$$

or $$x\frac{dy}{dx} = \frac{dy}{dz} \quad , \left(x\frac{d}{dx} \equiv \frac{d}{dz}\right)$$

Therefore $$x\frac{d}{dx}\left(x\frac{dy}{dx}\right) = \frac{d}{dz}\left(\frac{dy}{dz}\right)$$

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} = \frac{d^2y}{dz^2}$$

$$x^2\frac{d^2y}{dx^2} = \frac{d^2y}{dz^2} - x\frac{dy}{dx}$$

$$= \frac{d^2y}{dz^2} - \frac{dy}{dz}$$

The given diff. equation when we substitute

$$x^2\frac{d^2y}{dx^2} = \frac{d^2y}{dz^2} - \frac{dy}{dz} \text{ , } x\frac{dy}{dx} = \frac{dy}{dz}$$

becomes

$$\frac{d^2y}{dz^2} - \frac{dy}{dz} - 3\frac{dy}{dz} + 4y = 0$$

This is an equation with constant coefficients. $\frac{d^2y}{dz^2} - 4\frac{dy}{dz} + 4y = 0$

Assume $y = e^{rz}$ as a solution

The characteristic equation is $r^2 - 4r + 4 = 0$

$$(r-2)^2 = 0$$

$r = 2$ is a double root of the characteristic equation.

Therefore $$y = C_1e^{2z} + C_2ze^{2z}$$

$$= C_1(e^z)^2 + C_2(\ln x)(e^z)^2$$

That is $y(x) = C_1x^2 + C_2(\ln x)x^2$ is the general solution of the given diff. equation.

Substitution gives:

$$x^m m(m-1)(m-2) - 2x^m m(m-1) + 13x^m m - 13x^m = 0$$

Thus, $x^m(m^3 - 5m^2 + 17m - 13) = 0$

Or, $x^m(m-1)(m^2 - 4m + 13) = 0$

Since $x > 0$, we have $m - 1 = 0$ and $m^2 - 4m + 13 = 0$

and the roots are $m_1 = 1$, $m_2 = 2 + 3i$, $m_3 = 2 - 3i$

So the general solution is:

$$y = c_1 x + x^2(c_2 \cos(3 \ln x) + c_3 \sin(3 \ln x))$$

2-26

Find the general solution of $4x^2y'' - 8xy' + 9y = 0$.

****

In this Euler Equation, we use a trial solution $y = x^r$, whose derivatives are $y' = rx^{r-1}$ and $y'' = r(r-1)x^{r-2}$. Substituting into the equation, we get

$$4(r^2 - r) - 8r + 9 = 0,$$

or $4r^2 - 12r + 9 = 0$. This quadratic equation has only one (repeated)

Let $x = e^t$, or $t = \ln x$.

Then note $x \cdot y' \equiv x \cdot \frac{dy}{dx} = \frac{dy}{dt}$, and $x^2 \cdot y'' = \left(\frac{d^2y}{dt^2} - \frac{dy}{dt}\right)$.

The given differential equation now possesses constant coefficients in the y-t plane; that is, the equation now reads:

$$\left[3 \cdot \left(\frac{d^2y}{dt^2} - \frac{dy}{dt}\right) + 2 \cdot \left(\frac{dy}{dt}\right)\right] = 0 \quad , \text{ or}$$

$$(3 \cdot D^2 - D)\{y(t)\} = 0 \; ; \qquad D\{\,\} \equiv \frac{d}{dt}\{\,\}.$$

Thus $\left[D\left(D - \frac{1}{3}\right)\right]\{y(t)\} = 0$ gives characteristic roots $D = 0$, $D = \frac{1}{3}$. Then, the general solution to the given differential equation reads

$$y = y(t) = a_1 \cdot e^{0 \cdot t} + a_2 \cdot e^{\frac{1}{3}t} = a_1 + a_2 \cdot (e^t)^{\frac{1}{3}} \quad , \text{ or}$$

$$y = y(x) = a_1 + a_2 \cdot x^{\frac{1}{3}}.$$

Note herein that a_1, a_2, r_1, r_2 = constants.

2-25

Solve the Cauchy-Euler equation:

$$x^3y''' - 2x^2y'' + 13xy' - 13y = 0, \qquad x > 0$$

****

We assume that $y = x^m$. Then $y' = m\,x^{m-1}$, $y'' = m(m-1)\,x^{m-2}$ and $y''' = m(m-1)(m-2)\,x^{m-3}$.

$$y = c_1 x^{1/2} + c_2 x^{-5}.$$

2-28

Solve the equation $x^2 \frac{d^2y}{dx^2} - 3x \frac{dy}{dx} + 5y = 0$. (1)

* *

Assume a solution of the form $y = x^m$; then

$$\frac{dy}{dx} = m x^{m-1}, \quad \frac{d^2y}{dx^2} = m(m-1) x^{m-2}.$$

Substituting these into equation (1), we have

$$x^m [\, m(m-1) - 3m + 5 \,] = 0$$

so that

$$x^m (m^2 - 4m + 5) = 0.$$

Thus $y = x^m$ will be a solution of (1), if m is a root of the auxiliary equation

$$m^2 - 4m + 5 = 0.$$

Using the quadratic formula, we find

$$m = 2 \pm i.$$

Therefore the general solution is given by

$$y = x^2 [\, c_1 \cos(\ln x) + c_2 \sin(\ln x) \,],$$

where c_1 and c_2 are arbitrary constants.

real root $r = 3/2$. Hence the pair of independent solutions are $y_1 = x^{3/2}$ and $y_2 = x^{3/2} \cdot \ln x$. Thus, the general solution is $y = x^{3/2}(c_1 + c_2 \ln x)$.

2-27

Find the general solution:

$$2x^2 \frac{d^2y}{dx^2} + 11x \frac{dy}{dx} - 5y = 0$$

****

THIS IS A CAUCHY-EULER EQUATION, SO WE SET $x = e^t$ AND APPLY THE EQUATIONS

$$\begin{cases} \frac{dy}{dx} = \frac{1}{x} \frac{dy}{dt} \\ \frac{d^2y}{dx^2} = \frac{1}{x^2}\left(\frac{d^2y}{dt^2} - \frac{dy}{dt}\right). \end{cases}$$

SUBSTITUTING INTO THE ORIGINAL EQUATION AND CANCELLING THE x'S, WE OBTAIN AN EQUATION WITH CONSTANT COEFFICIENTS:

$$2\left(\frac{d^2y}{dt^2} - \frac{dy}{dt}\right) + 11\left(\frac{dy}{dt}\right) - 5y = 0$$

$$2\frac{d^2y}{dt^2} + 9\frac{dy}{dt} - 5y = 0$$

THE CHARACTERISTIC EQUATION $2m^2 + 9m - 5 = (2m-1)(m+5) = 0$ HAS ROOTS $m_1 = 1/2$ AND $m_2 = -5$.

THE GENERAL SOLUTION, IN TERMS OF t, IS GIVEN BY

$$y = c_1 e^{\frac{1}{2}t} + c_2 e^{-5t}.$$

SINCE $e^{\frac{1}{2}t} = (e^t)^{1/2} = x^{1/2}$ AND $e^{-5t} = (e^t)^{-5} = x^{-5}$, THE GENERAL SOLUTION, IN TERMS OF x, BECOMES

2-30

Solve $x^2y'' - 3xy' + 3y = 0$

****

Assume $y = x^r$ as a solution of the equation
Then $y' = r x^{r-1}$, $y'' = r(r-1)x^{r-2}$ and
substitute for y'', y' and y in the given eq.

$$x^2(r(r-1)x^{r-2}) - 3x(r x^{r-1}) + 3(x^r) = 0$$

That is $$x^r\{r(r-1) - 3r + 3\} = 0$$

The characteristic equation is

$$r^2 - 4r + 3 = 0$$
$$(r-1)(r-3) = 0$$
$$r = 1, 3$$

Therefore the general solution of the given Euler differential equation is

$$y(x) = C_1 x + C_2 x^3$$

Alternatively Euler's equation can be solved by using the substitution $x = e^z$.

2-31

Obtain the general solution of the Euler DE

$$x^2y'' - 2y = 0.$$

****

The Euler DE may be solved by the substitution

$$y = x^r$$

Then $y' = rx^{r-1}$, $y'' = r(r-1)x^{r-2}$

2-29

Solve the initial value problem $4x^2y'' + y = 0$, $y(1)=2$, $y'(1)=-1$.

****

an Euler equation, so seek solutions of the form x^m

let $y = x^m$, so $y' = mx^{m-1}$ and $y'' = m(m-1)x^{m-2}$

into the de: $4x^2m(m-1)x^{m-2} + x^m = 0$

$4m(m-1)x^m + x^m = 0$

$(4m^2 - 4m + 1)x^m = 0$

$(4m^2 - 4m + 1) = 0$

$(2m-1)^2 = 0$

$m = \frac{1}{2}$, double root

general solution: $y = c_1 x^{1/2} + c_2 x^{1/2} \ln x$

$y(1)=2$: $2 = c_1 + 0$ $\quad c_1 = 2$

so $y = 2x^{1/2} + c_2 x^{1/2} \ln x$

$y'(1)=-1$: $y' = x^{-1/2} + c_2 x^{1/2}\left(\frac{1}{x}\right) + c_2 \ln x \left(\frac{1}{2} x^{-1/2}\right)$

$= x^{-1/2} + c_2 x^{-1/2} + \frac{1}{2} c_2 x^{-1/2} \ln x$

$-1 = 1 + c_2 + 0$ $\quad c_2 = -2$

thus $y = 2x^{1/2} - 2x^{1/2} \ln x$

2-33

Solve for all positive x; $x^3y''' - 4x^2y'' + 10xy' - 12y = 0.$

****

THIS CAN BE SOLVED, PAINFULLY, WITH AN INFINITE SERIES. HOWEVER, IF IT IS RECOGNIZED AS AN "EULER" OR "CAUCHY-EULER" DIFFERENTIAL EQUATION (IN EACH TERM, THE POWER OF X IS THE SAME AS THE ORDER OF THE DERIVATIVE) THEN THE CHANGE OF INDEPENDENT VARIABLE $X = e^t$ OR $t = \ln x$ WILL PRODUCE A NEW EQUATION IN y AND t WHICH IS CONSTANT COEFFICIENT. WE HAVE, WITH $D_x = \frac{d}{dx}$ AND $D_t = \frac{d}{dt}$,

$$xD_x \sim D_t, \quad x^2D_x^2 \sim D_t(D_t-1), \quad x^3D_x^3 \sim D_t(D_t-1)(D_t-2)$$

AND, IN GENERAL, $x^nD_x^n \sim (D_t)(D_t-1)(D_t-2)\cdots(D_t-n+1)$.

THE GIVEN EQUATION CAN BE WRITTEN IN THE FORM

$$(x^3D_x^3 - 4x^2D_x^2 + 10xD_x - 12)y = 0. \quad \text{WITH } t = \ln x,$$

$$[D_t(D_t-1)(D_t-2) - 4D_t(D_t-1) + 10D_t - 12]y = 0 \quad \text{OR}$$

$$(D_t^3 - 7D_t^2 + 16D_t - 12)y = 0.$$

USING SYNTHETIC DIVISION,

$$\begin{array}{c|cccc} 2 & 1 & -7 & 16 & -12 \\ & & 2 & -10 & 0 \\ \hline & 1 & -5 & 6 & \end{array} \qquad \begin{array}{c|ccc} 2 & 1 & -5 & 6 \\ & & 2 & -6 \\ \hline & 1 & -3 & 0 \end{array}$$

$$D_t^3 - 7D_t^2 + 16D_t - 12 = (D_t-2)(D_t-2)(D_t-3).$$

$$\therefore y = c_1e^{3t} + c_2e^{2t} + c_3te^{2t}.$$

BUT $t = \ln x$, SO $e^{3t} = e^{3\ln x} = e^{\ln x^3} = x^3$. $e^{2t} = e^{\ln x^2} = x^2$.

$$\therefore y = c_1x^3 + c_2x^2 + c_3x^2\ln x.$$

Substituting into the DE:

$x^2y'' - 2y = x^2[r(r-1)x^{r-2}] - 2x^r = 0$

If $x \neq 0$, $r(r-1) - 2 = 0$.

$\Rightarrow r^2 - r - 2 = (r-2)(r+1) = 0.$

Hence $r = -1, 2$

General solution $\boxed{y = c_1 x^{-1} + c_2 x^2}$

2-32

Solve $x^2y'' + 2xy' - 6y = 0$ where $y(1) = 1$, $y'(1) = -1$.

****

This is an Euler equation with auxiliary equation $m^2 + m - 6 = 0$

$m = 2, m = -3.$

The solution is $y = c_1 x^2 + c_2 x^{-3}$.

$y' = 2c_1 x - 3c_2 x^{-4}$

We need to solve

$$\left\{ \begin{array}{l} c_1 + c_2 = 1 \\ 2c_1 - 3c_2 = -1 \end{array} \right\}$$

We get $c_1 = 2/5$, $c_2 = 3/5$

$$y = \frac{2x^2}{5} + \frac{3}{5x^3}$$

2-35

Solve the differential equations indicated.

(a) $t^2y'' + ty' + 4y = 0, \quad t > 0$

(b) $t^3y''' + 4t^2y'' - 8ty' + 8y = 0, \quad t > 0$

****

The transformation $t = e^s$ or $\ln t = s$ will generate differential equations with constant coefficients

Note: $y' = \frac{dy}{dt} = \frac{dy}{ds}\frac{ds}{dt} = \frac{dy}{ds}\cdot\frac{1}{t}$ (chain rule)

$$y'' = \frac{d^2y}{dt^2} = \frac{d}{dt}\left[\frac{dy}{ds}\cdot\frac{1}{t}\right] = \frac{d}{dt}\left[\frac{dy}{ds}\right]\cdot\frac{1}{t} + \frac{dy}{ds}\cdot\left(-\frac{1}{t^2}\right)$$

$$= \frac{d}{ds}\left[\frac{dy}{ds}\right]\cdot\frac{ds}{dt}\cdot\frac{1}{t} - \frac{1}{t^2}\frac{dy}{ds} \quad \text{(chain rule)}$$

$$y'' = \frac{d^2y}{ds^2}\cdot\frac{1}{t^2} - \frac{1}{t^2}\frac{dy}{ds} = \frac{1}{t^2}\left[\frac{d^2y}{ds^2} - \frac{dy}{ds}\right]$$

Similarly: $\frac{d^3y}{dt^3} = \frac{d}{dt}\left[\frac{1}{t^2}\cdot\left\{\frac{d^2y}{ds^2} - \frac{dy}{ds}\right\}\right] = -2t^{-3}\{\} + \frac{1}{t^2}\cdot\frac{d}{dt}\{\}$

$$= -2t^{-3}\{\} + \frac{1}{t^2}\frac{d}{ds}\{\}\frac{ds}{dt} \quad \text{(chain rule)}$$

$$= -2t^{-3}\left\{\frac{d^2y}{ds^2} - \frac{dy}{ds}\right\} + \frac{1}{t^2}\left\{\frac{d^3y}{ds^3} - \frac{d^2y}{ds^2}\right\}\cdot\frac{1}{t}$$

$$y''' = \frac{d^3y}{dt^3} = \frac{1}{t^3}\frac{d^3y}{ds^3} - \frac{3}{t^3}\frac{d^2y}{ds^2} + \frac{2}{t^3}\frac{dy}{ds} = \frac{1}{t^3}\left[\frac{d^3y}{ds^3} - 3\frac{d^2y}{ds^2} + 2\frac{dy}{ds}\right]$$

(a) becomes: $t^2\cdot\frac{1}{t^2}\left[\frac{d^2y}{ds^2} - \frac{dy}{ds}\right] + t\cdot\frac{dy}{ds}\cdot\frac{1}{t} + 4y = 0$

$$\Rightarrow \frac{d^2y}{ds^2} + 0\cdot\frac{dy}{ds} + 4y = 0 \Rightarrow r^2 + 4 = 0 \Rightarrow r = \pm 2i$$

$\Rightarrow$ Solutions: $\cos 2s, \sin 2s$

$\therefore$ the general solution is $y = c_1\cos 2s + c_2\sin 2s$, but $s = \ln t$, $\therefore$ $y = c_1\cos 2(\ln t) + c_2\sin 2(\ln t)$ is the general solution of the original differential equation.

(b) becomes: $t^3\cdot\frac{1}{t^3}\left[\frac{d^3y}{ds^3} - 3\frac{d^2y}{ds^2} + 2\frac{dy}{ds}\right] + 4t^2\cdot\frac{1}{t^2}\left[\frac{d^2y}{ds^2} - \frac{dy}{ds}\right] - 8t\cdot\frac{1}{t}\frac{dy}{ds} + 8y = 0$

$$\Rightarrow \frac{d^3y}{ds^3} + \frac{d^2y}{ds^2} - 10\frac{dy}{ds} + 8y = 0$$

Hence, the characteristic equation

$\Rightarrow r^3 + r^2 - 10r + 8 = 0$ Observe $r = 1$ is a solution; $\therefore$ $(r-1)$ is a factor, hence

$\therefore (r-1)[r^2 + 2r - 8] = (r-1)(r+4)(r-2) = 0 \Rightarrow r = 1, r = -4$ and $r = 2$

2-34

Find the general solution: $x^3\frac{d^3y}{dx^3} + 3x^2\frac{d^2y}{dx^2} - 6x\frac{dy}{dx} + 6y = 0$

****

THIS IS A CAUCHY-EULER EQUATION, SO USING THE SUBSTITUTION $x = e^t$ (OR $\ln x = t$) WE OBTAIN

$$\frac{dy}{dx} = \frac{1}{x}\frac{dy}{dt}$$

$$\frac{d^2y}{dx^2} = \frac{1}{x^2}\left(\frac{d^2y}{dt^2} - \frac{dy}{dt}\right)$$

$$\frac{d^3y}{dx^3} = \frac{1}{x^3}\left(\frac{d^3y}{dt^3} - 3\frac{d^2y}{dt^2} + 2\frac{dy}{dt}\right)$$

SUBSTITUTING INTO THE ORIGINAL EQUATION AND CANCELLING THE VARIOUS POWERS OF x, WE HAVE AN EQUATION WITH CONSTANT COEFFICIENTS:

$$\left(\frac{d^3y}{dt^3} - 3\frac{d^2y}{dt^2} + 2\frac{dy}{dt}\right) + 3\left(\frac{d^2y}{dt^2} - \frac{dy}{dt}\right) - 6\left(\frac{dy}{dt}\right) + 6y = 0$$

$$\frac{d^3y}{dt^3} - 7\frac{dy}{dt} + 6y = 0$$

THE CHARACTERISTIC EQUATION IS $m^3 - 7m + 6 = 0$. ONE OBVIOUS ROOT IS $m_1 = 1$, SO WE DIVIDE BY THE FACTOR $(m-1)$ TO OBTAIN THE OTHER TWO ROOTS:

$$(m-1)(m^2 + m - 6) = 0$$

$$(m-1)(m-2)(m+3) = 0$$

THE THREE ROOTS ARE $m_1 = 1$, $m_2 = 2$, AND $m_3 = -3$. THUS THE GENERAL SOLUTION, IN TERMS OF t, IS GIVEN BY

$$y = c_1 e^t + c_2 e^{2t} + c_3 e^{-3t}.$$

CONVERTING BACK TO x, WE NOTE THAT $x = e^t$ IMPLIES THAT $e^{2t} = (e^t)^2 = x^2$ AND $e^{-3t} = (e^t)^{-3} = x^{-3}$.

GENERAL SOLUTION: $y = c_1 x + c_2 x^2 + c_3 x^{-3}$

2-37

Solve $x^2y'' + 2xy' - 2y = 0$.

****

We recognize this as an Euler Eq. - i.e. it is of the form $x^2y'' + axy' + by = 0$ with $a = 2$ and $b = -2$. The change of independent variable $x = e^t$ $(t = \ln x)$ changes this DE to the constant coefficient equation $\frac{d^2y}{dt^2} + (a-1)\frac{dy}{dt} + by = 0$. For our eq. this becomes $\frac{d^2y}{dt^2} + \frac{dy}{dt} - 2y = 0$. The solution to the auxiliary quadratic eq. $r^2 + r - 2 = 0$ is $r = -2, 1$. The general solution to the DE in terms of t is $y = C_1e^{-2t} + C_2e^{t}$. Substituting $t = \ln x$ we get

$$y = C_1e^{-2\ln x} + C_2e^{\ln x} = C_1e^{\ln x^{-2}} + C_2e^{\ln x}$$

$$y = C_1x^{-2} + C_2x \quad \text{or}$$

$$y = \frac{C_1}{x^2} + C_2x.$$

are solutions of the characteristic equation

$\therefore\ y = c_1 e^{s} + c_2 e^{2s} + c_3 e^{-4s}$, but $s = \ln t$

$\therefore\ y = c_1 e^{\ln t} + c_2 e^{2\ln t} + c_3 e^{-4\ln t}$

$\therefore\ y = c_1 t + c_2 t^2 + c_3 t^{-4}$ is the general solution of the original differential equation.

2-36

Find the general solution of

$$x^2y'' + 3xy' + y = 0, \quad x > 0 .$$

****

set $z = \ln x$, $\frac{dz}{dx} = \frac{1}{x}$, $x = e^z$, $\frac{1}{x} = e^{-z}$

$$y' = \frac{dy}{dz}\frac{dz}{dx} = \frac{1}{x}\frac{dy}{dz}$$

$$y'' = \frac{1}{x^2}\left(\frac{d^2y}{dz^2} - \frac{dy}{dz}\right)$$

so

$$x^2y'' + 3xy' + y = \frac{d^2y}{dz^2} - \frac{dy}{dz} + 3\frac{dy}{dz} + y$$

$$= \frac{d^2y}{dz^2} + 2\frac{dy}{dz} + y = 0.$$

$$r^2 + 2r + 1 = (r+1)^2 = 0$$

$$y = (c_1 + c_2 z)e^{-z}$$

$$y(x) = (c_1 + c_2 \ln x)\frac{1}{x} .$$

2-39

Check $y_1(x)=x+1$ is a solution of

$$xy'' - (x+1)y' + y = 0 \qquad (1)$$

then find the general solution of (1).

****

Set $y_2(x) = v(x)\,y_1(x)$, we want $y_2(x)$ is also a solution of (1), so

$$0 = x y_2'' - (x+1)\,y_2' + y_2 = x(x+1)\,v'' + [2x - (x+1)^2]\,v'$$

Set $v' = w$

$$x(x+1)\,w' = [(x+1)^2 - 2x]\,w$$

$$\frac{w'}{w} = 1 + \frac{1}{x} - \frac{2}{x+1} \quad , \quad w(x) = \frac{x}{(1+x)^2}\,e^x$$

$$v' = w = \frac{x}{(1+x)^2}e^x = e^x\left[\frac{1}{1+x} - \frac{1}{(1+x)^2}\right] = \left(\frac{e^x}{1+x}\right)'$$

So

$$v(x) = \frac{e^x}{1+x}$$

$$y_2(x) = e^x$$

$$\begin{vmatrix} x+1 & e^x \\ 1 & e^x \end{vmatrix} = xe^x \neq 0 \qquad (x \neq 0)$$

so $x+1$, e^x are linearly independent. Therefore the general solution of (1) is

$$y(x) = C_1(x+1) + C_2 e^x .$$

REDUCTION OF ORDER

2-38

Solve the second order differential equation

$$y'' + y = \sec^3 x$$

****

The characteristic equation of the homogeneous equation is $r^2+1=0$, and that part of the solution is $c_1 \cos x + c_2 \sin x$. Since the equation is second order, both variation of parameters and reduction of order methods will solve the equation. Since one solution of the homogeneous equation, $\cos x$, is related to $\sec^3 x$, reduction of order may be faster.

Reduction of order. Set $y = u\cos x$.

$y' = u'\cos x - u\sin x$, and $y'' = u''\cos x - 2u'\sin x - u\cos x$,

so $y''+y = u''\cos x - 2u'\sin x = \sec^3 x$, or

$$u'' + (-2\tan x)u' = \sec^4 x.$$

The integrating factor of this first order equation in u' is $e^{\int -2\tan x\,dx} = e^{2\ln|\cos x|} = \cos^2 x$, giving

$$(u'\cos^2 x)' = (\cos^2 x)u'' - 2(\cos x \sin x)u' = \sec^2 x.$$

This gives $u'\cos^2 x = \tan x + c_1$, and

$u' = \tan x \sec^2 x + c_1 \sec^2 x$, so $u = \int(\tan x \sec^2 x + c_1 \sec^2 x)dx$

$u = \frac{1}{2}\tan^2 x + c_1 \tan x + c_2$, and $y = u\cos x$ gives

$$y = \frac{1}{2}\tan^2 x \cos x + c_1 \sin x + c_2 \cos x.$$

Note: $\frac{1}{2}\tan^2 x \cos x = \frac{1}{2}\frac{\sin^2 x}{\cos x} = \frac{1}{2}\frac{1-\cos^2 x}{\cos x} = \frac{1}{2}\sec x - \frac{1}{2}\cos x$,

so $y = \frac{1}{2}\sec x + c_1 \sin x + (c_2 - \frac{1}{2})\cos x.$

2-41

Given that $y = 1/x$ is a solution of the equation

$$x^2y'' + 3xy' + y = 0. \tag{1}$$

find a second linearly independent solution.

* *

Rewrite the differential equation as

$$y'' + \frac{3}{x}y' + \frac{1}{x^2}y = 0 \tag{2}$$

which is of the form

$$y'' + p(x)y' + q(x)y = 0. \tag{3}$$

Here $p(x) = \frac{3}{x}$, $q(x) = \frac{1}{x^2}$.

If $y_1(x)$ is a solution of equation (3), then by the method of reduction of order, a second solution is given by

$$y_2(x) = y_1(x) \int \frac{1}{y_1^2(x)} e^{-\int p(x)dx} dx.$$

In the present case, with $y_1(x) = 1/x$, we obtain

$$y_2(x) = \frac{1}{x} \int \frac{1}{(1/x)^2} e^{-\int \frac{3}{x} dx} dx$$

$$= \frac{1}{x} \int x^2 e^{-3\ln x} dx$$

2-40

If $y_1 = e^t$ is a solution of $ty'' - (t+1)y' + y = 0$, $t > 0$, find the general solution for $t > 0$.

****

We need to find a 2nd, linearly independent solution y_2. Set $y_2 = y_1 \cdot w(t) = e^t w(t)$

$$y_2' = e^t(w' + w)$$
$$y_2'' = e^t(w'' + w') + e^t(w' + w)$$
$$\therefore y_2'' = e^t(w'' + 2w' + w)$$

If y_2 is to be a solution, then it must satisfy the d.e.: $t \cdot e^t(w'' + 2w' + w) - (t+1)e^t(w' + w) + e^t w(t) = 0$

$$\Rightarrow t w'' + (2t - t - 1)w' + 0 \cdot w = 0$$
$$\therefore t w'' + (t-1)w' = 0 \quad \text{or} \quad w'' + \left(1 - \frac{1}{t}\right)w' = 0 \quad (*)$$

Set $\left.\begin{aligned} v &= w' \\ v' &= w'' \end{aligned}\right\} \Rightarrow (*)$ becomes: $\underline{v' + \left(1 - \frac{1}{t}\right)v = 0}$ This is a first order, linear d.e.

$\therefore$ The integrating factor is $e^{\int (1 - \frac{1}{t})dt} = e^{t - \ln t} = e^t e^{-\ln t} = e^t \cdot e^{\ln t^{-1}} = \frac{1}{t} e^t$

Multiply by the integrating factor:

$$\frac{e^t}{t}\left[v' + \left(1 - \frac{1}{t}\right)v\right] = 0 \cdot \frac{e^t}{t} = 0$$

$$\therefore \frac{d}{dt}\left[\frac{e^t}{t} \cdot v\right] = 0 \Rightarrow \frac{e^t}{t} v = c \Rightarrow v = c t e^{-t}$$

or, without any loss of generality, $c = 1$. $\therefore \underline{v = t e^{-t}}$

$$\therefore w'(t) = v(t) = t e^{-t} \Rightarrow w(t) = \int t e^{-t} dt = -t e^{-t} + \int e^{-t} dt$$

$\{u = t \quad dv = e^{-t}dt\}$
$\{du = dt \quad v = -e^{-t}\}$

$$= -t e^{-t} - e^{-t} = e^{-t}(-1 - t)$$
$$\therefore y_2 = e^t w = e^t \cdot e^{-t}(-1 - t) = -1 - t$$
$$y_2 = -1 - t$$

$\therefore$ The general solution is $\underline{y = c_1 e^t + c_2(-1 - t)}$.

(Note: could have used $y_2 = 1 + t$ or any other convenient multiple.)

Then applying the formula

$$v = \int \frac{1}{y_1^2} e^{-\int P\,dx}\,dx$$

we find $v = \sin x$ and $y_2 = x \sin x$. Therefore the general solution of (1) is $c_1 x + c_2 x \sin x$.

2-43

Solve: $xy'' - 2(x+1)y' + (x+2)y = 0$,
given that $y_1 = e^x$ is a solution.

****

In standard form, the equation becomes

$$y'' + \left(-2 - \frac{2}{x}\right)y' + \left(1 + \frac{2}{x}\right)y = 0$$

Thus $P(x) = -2 - 2/x$ and $-\int P(x)\,dx = 2x + \ln x^2$

The reduction of order formula $y_2 = y_1 \int \frac{e^{-\int P(x)dx}}{(y_1)^2}\,dx$ gives

$$y_2 = e^x \int \frac{e^{2x + \ln x^2}}{e^{2x}}\,dx$$

$$= e^x \int x^2\,dx = \frac{1}{3} x^3 e^x$$

Thus, the general solution is: $y = c_1 e^x + c_2 x^3 e^x$

$$= \frac{1}{x}\int x^2 \cdot \frac{1}{x^3}\,dx = \frac{1}{x}\int \frac{1}{x}\,dx = \frac{1}{x}\ln x .$$

The two solutions $y_1(x) = \frac{1}{x}$ and $y_2 = \frac{1}{x}\ln x$ are linearly independent since their Wronskian

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

$$= \begin{vmatrix} \frac{1}{x} & \frac{1}{x}\ln x \\ -\frac{1}{x^2} & \frac{1}{x^2}(1-\ln x) \end{vmatrix} = \frac{1}{x^3}$$

is not zero.

2-42

Find the general solution of the equation

$$xy'' - (2 - x\tan x)y' + (2/x - \tan x)y = 0$$

(Hint: $y=x$ is a solution.)

****

Write the equation in the form

$$y'' - \frac{2 - x\tan x}{x}y' + \frac{\frac{2}{x} - \tan x}{x}y = 0$$

Assume that the second solution is

$$y_2 = vy_1 = vx$$

2-45

$f(x) = e^x$ is a solution of $(x-1)y'' - xy' + y = 0$. Determine a second linearly independent solution.

****

Let $y = v(x)f(x) = ve^x$. Then

$y' = v'e^x + ve^x$

and $y'' = v''e^x + 2v'e^x + ve^x$

Substituting these in the differential equation, we have

$$(x-1)(v''e^x + 2v'e^x + ve^x) - x(v'e^x + ve^x) + ve^x = 0,$$

which simplifies to

$$(x-1)v'' + (x-2)v' = 0.$$

Let $w = v'$. Then $w' + \frac{x-2}{x-1}w = 0$, $(x \neq 1)$.

$$I(x) = \exp\left(\int \tfrac{x-2}{x-1}\,dx\right)$$
$$= \exp\left(\int (1 - \tfrac{1}{x-1})\,dx\right)$$
$$= \exp[x - \ln(x-1)]$$
$$= \frac{e^x}{x-1}$$ is an integrating factor.

Thus $\frac{d}{dx}\left(\frac{e^x}{x-1}w\right) = 0$ and $w = c_1(x-1)e^{-x}$,

i.e. $v' = c_1(x-1)e^{-x}$.

Integrating this by parts, we obtain

$$v = -c_1xe^{-x} + c_2.$$

Thus $y = (-c_1xe^{-x} + c_2)e^x$

$$= -c_1x + c_2e^x$$

is the general solution, c_1 and c_2 arbitrary, where $x \neq 1$. But this solution satisfies the D.E. when $x = 1$. Therefore $y = -c_1x + c_2e^x$ is the general solution. Taking $c_1 = -1$ and $c_2 = 0$, we obtain a second solution, $y = x$.

2-44

Consider $2x^2y'' + 3xy' - y = 0$ on the interval $0 < x < \infty$.

a. Show that $y_1 = \sqrt{x}$ is a solution of the D. E.

b. Find a second linearly independent solution, y_2. (As a check on your work, verify that your y_2 is really a solution of the D. E.)

c. Using the given solution y_1 and your y_2, compute the Wronskian, $W(y_1, y_2)(x)$.

d. Give the general solution of the D. E.

****

a. $y_1 = x^{1/2}$; $y_1' = \frac{1}{2}x^{-1/2}$; $y_1'' = -\frac{1}{4}x^{-3/2}$.

$$2x^2(-\tfrac{1}{4}x^{-3/2}) + 3x(\tfrac{1}{2}x^{-1/2}) - x^{1/2} =$$
$$-\tfrac{1}{2}x^{1/2} + \tfrac{3}{2}x^{1/2} - x^{1/2} = (-\tfrac{1}{2} + \tfrac{3}{2} - 1)x^{1/2} = 0.$$

b. Reduction of order. The standard form for the D.E. is $y'' + py' + qy = 0$. Here $p(x) = 3x/2x^2 = 3/2x$ [and not $3x$] and $q(x) = -1/2x^2$ [and not -1].

Let $y = v(x)y_1(x)$. $v(x) = \int^x \frac{1}{(y_1(t))^2} \exp[-\int^t p(s)\,ds]\,dt$

$$= \int^x \frac{1}{(\sqrt{t})^2} \exp[-\int^t \tfrac{3}{2}\,\tfrac{ds}{s}]\,dt = \int^x \frac{1}{t} \exp(-\tfrac{3}{2}\ln t)\,dt$$
$$= \int^x \frac{1}{t} t^{-3/2}\,dt = \int^x t^{-5/2}\,dt = -\tfrac{2}{3}t^{-3/2}.$$

Take $v(x) = x^{-3/2}$; $y_2(x) = x^{1/2}x^{-3/2} = x^{-1} = \frac{1}{x}$

c. $W(x^{1/2}, \frac{1}{x}) = \begin{vmatrix} x^{1/2} & x^{-1} \\ \frac{1}{2}x^{-1/2} & -x^{-2} \end{vmatrix} = -x^{-3/2} - \frac{1}{2}x^{-3/2}$

$$= -\tfrac{3}{2}x^{-3/2}$$

d. $y = C_1\sqrt{x} + C_2/x$.

2-47

Given that $y_1 = 1/x$ is a solution of

$$xy'' + (2 + x)y' + y = 0, \quad x > 0,$$

use reduction of order to find the general solution.

****

Method 1 Since $y_1 = 1/x$ already satisfies the DE, look for another solution of the form $y_2 = \frac{1}{x}v$, then, $y_2' = \frac{1}{x}v' - \frac{1}{x^2}v$

$$y_2'' = \frac{1}{x}v'' - \frac{1}{x^2}v' - \frac{1}{x^2}v' + \frac{2}{x^3}v$$

$$x y_2'' + (2+x) y_2' + y_2$$

$$= x\left(\frac{1}{x}v'' - \frac{2}{x^2}v' + \frac{2v}{x^3}\right) + (2+x)\left(\frac{1}{x}v' - \frac{1}{x^2}v\right) + \frac{1}{x}v = 0$$

Collect terms involving v'', v' and v:

$$v'' + \left(-\frac{2}{x} + \frac{2+x}{x}\right)v' + \left(\frac{2}{x^2} - \frac{2+x}{x^2} + \frac{1}{x}\right)v = 0$$

Simplifying: $v'' + v' = 0$

This is an equation with the dependant variable missing.

Set $v' = w$, $v'' = w'$

$$w' + w = 0$$

Linear for $w(x)$, integrating factor $\mu(x) = e^{\int 1\,dx} = e^x$

2-46

Find the general solution to

$$y'' - \frac{1}{x}y' + 4x^2y = 0,$$

given that $y = \cos(x^2)$ is a solution.

****

Reduction of order: $y = u\cos(x^2)$.

$y' = u'\cos(x^2) + u\cdot(-2x\sin(x^2))$

$y'' = u''\cos(x^2) + 2u'(-2x\sin(x^2)) + u(-2\sin(x^2) - 4x^2\cos(x^2))$. So:

$y'' = u''\cos(x^2) - 4xu'\sin(x^2) + u(-2\sin(x^2) - 4x^2\cos(x^2))$

$-\frac{1}{x}y' = -\frac{1}{x}u'\cos(x^2) + 2u\sin(x^2)$

$4x^2y = 4x^2u\cos(x^2)$

and $u''\cos(x^2) + (-4x\sin(x^2) - \frac{1}{x}\cos(x^2))u' = 0$

or $u'' + (-4x\tan(x^2) - \frac{1}{x})u' = 0$. The integrating factor for this first order equation in u' is

$e^{\int(-4x\tan(x^2) - \frac{1}{x})dx}$. $\int -4x\tan(x^2)dx = \int -2\tan v\,dv = 2\ln|\cos v|$

with $v = x^2$ substitution, so the integrating factor is

$e^{2\ln|\cos(x^2)| - \ln|x|}$ or $\cos^2(x^2)/x$.

$(u'\cos^2(x^2)/x)' = 0$ gives $u' = c_1 x\sec^2(x^2)$ and

$u = \int c_1 x\sec^2(x^2)dx = \frac{1}{2}c_1\int \sec^2 v\,dv = \frac{1}{2}c_1\tan(x^2) + c_2$

(same substitution, $v = x^2$), so

$$y = u\cos(x^2) = \frac{1}{2}c_1\tan(x^2)\cos(x^2) + c_2\cos(x^2)$$
$$= \frac{1}{2}c_1\sin(x^2) + c_2\cos(x^2).$$

2-48

Find the general solution to $x^2y'' - xy' - 3y = 0$ on $x > 0$ given that $y_1 = x^3$ is a solution.

****

If y_1 is a solution to the DE $y''+p(x)y'+q(x)y=0$, then a second linearly independent solution y_2 is given by $y_2 = vy_1$ where v satisfies $v' = \frac{1}{y_1^2} e^{-\int p(x)\,dx}$

We must put our eq. in this form - i.e. divide by x^2.

$y'' - \frac{1}{x}y' - \frac{3}{x^2}y = 0$, $x>0$. $p(x) = -\frac{1}{x}$ and $q(x) = -\frac{3}{x^2}$.

$-\int p(x)\,dx = \int \frac{1}{x}\,dx = \ln|x| = \ln x$ on $x>0$.

$v' = \frac{1}{(x^3)^2} e^{\ln x} = \frac{x}{x^6} = \frac{1}{x^5}$ Integrating gives $v = \int x^{-5}dx = -\frac{1}{4}x^{-4}$. Therefore

$y_2 = vy_1 = (-\frac{1}{4}x^{-4})(x^3) = -\frac{1}{4}x^{-1} = -\frac{1}{4x}$

The general solution is then $y = c_1y_1 + c_2y_2$

So $y = c_1x^3 + c_2(-\frac{1}{4x})$. We can replace the arbitrary constant $-\frac{c_2}{4}$ by c_2 and

$$e^x(w'+w)=0$$

Exact derivative $\frac{d}{dx}(e^x w)=0$

Integrate $e^x w = k_1$

$w = k_1 e^{-x} = v'$

Integrate again $v = -k_1 e^{-x} + k_2$

Then $y_2 = \frac{1}{x} v = -\frac{k_1}{x} e^{-x} + \frac{k_2}{x}$.

Since $y_1 = 1/x$ is already a solution, take $k_2 = 0$
For simplicity, take $k_1 = -1$. Then $y_2 = \frac{1}{x} e^{-x}$.
General solution $y = c_1 y_1 + c_2 y_2$

$$\boxed{y = \frac{c_1}{x} + \frac{c_2}{x} e^{-x}}$$

Method 2 Alternatively, using Abel's formula,

$$y_2 = y_1 \int \frac{e^{-\int p\,dx}}{y_1^2}\,dx \text{ is a 2nd solution}$$

where y_1 satisfies $y'' + p(x)y' + q(x)y = 0$.
Here $p(x) = \frac{2+x}{x}$, $e^{-\int p\,dx} = e^{-\int (\frac{2}{x}+1)\,dx}$

$$= e^{-2\ln x - x} = x^2 e^{-x}$$

$$y_2 = \frac{1}{x}\int \frac{x^{-2}e^{-x}}{x^{-2}}\,dx = \frac{1}{x}\int e^{-x}\,dx = -\frac{e^{-x}}{x}.$$

Once again $\boxed{y = \frac{c_1}{x} + \frac{c_2}{x} e^{-x}}$

2-50

$$x\frac{d^2y}{dx^2} - (2x + 1)\frac{dy}{dx} - (3x + 1)y = 0$$

a) Show that $y_1 = e^{-x}$ is a solution to this equation.

b) Use the method of reduction of order to find a second solution y_2.

****

a) $y_1 = e^{-x}$, $y_1' = -e^{-x}$, $y_1'' = e^{-x}$

$$x(e^{-x}) - (2x+1)(-e^{-x}) - (3x+1)(e^{-x})$$
$$= [x + (2x+1) - (3x+1)]\,e^{-x} = 0 \cdot e^{-x} = 0 \quad \checkmark$$

b) METHOD I: $y = v \cdot y_1 = v \cdot e^{-x}$

$$y' = (v' - v)e^{-x}$$
$$y'' = (v'' - 2v' + v)e^{-x}$$

$$x(v'' - 2v' + v)e^{-x} - (2x+1)(v' - v)e^{-x} - (3x+1)ve^{-x} = 0$$
$$e^{-x}[xv'' - 2xv' + xv - 2xv' - v' + 2xv + v - 3xv - v] = 0$$

$$\Longrightarrow \quad xv'' - (4x+1)v' = 0$$

WE DEFINE $w = v'$, SO $v'' = \frac{dw}{dx}$ AND THIS BECOMES A SEPARABLE EQUATION:

$$x\frac{dw}{dx} - (4x+1)w = 0$$
$$\frac{1}{w}dw - \frac{4x+1}{x}dx = 0$$
$$\int \frac{1}{w}dw - \int\left(4 + \frac{1}{x}\right)dx = \int 0$$
$$\ln|w| - 4x - \ln|x| = k_1$$
$$\frac{w}{x}e^{-4x} = e^{\ln|w| - 4x - \ln|x|} = e^{k_1} = k_2$$

$$\Longrightarrow \quad \frac{dv}{dx} = w = k_2 x e^{4x}$$
$$\int dv = \int k_2 x e^{4x}\,dx$$
$$v = k_2\left(\frac{1}{4}x - \frac{1}{16}\right)e^{4x} = k(4x-1)e^{4x}$$

rewrite our solution as

$$y = c_1 x^3 + \frac{c_2}{x}$$

2-49

Given that $\cos x/\sqrt{x}$ is a solution of the equation

$$\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} + \left(1 - \frac{1}{4x^2}\right) y = 0,$$

find the general solution of the equation by the method of reduction of order.

****

The given equation is of the type $\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)\,y = 0$ where $P(x) = \frac{1}{x}$.

So the general solution is $y(x) = c_1 y_1(x) + c_2 y_1(x) \int \frac{e^{-\int P dx}}{[y_1(x)]^2}\,dx$

where $y_1(x) = \frac{\cos x}{\sqrt{x}}$ and $P(x) = \frac{1}{x}$

In this example

$$y(x) = c_1 \frac{\cos x}{\sqrt{x}} + c_2 \frac{\cos x}{\sqrt{x}} \int \frac{e^{-\int \frac{1}{x} dx}}{\left[\frac{\cos x}{\sqrt{x}}\right]^2}\,dx$$

$$= c_1 \frac{\cos x}{\sqrt{x}} + c_2 \frac{\cos x}{\sqrt{x}} \int \frac{x\, e^{-\ln|x|}}{\cos^2 x}\,dx$$

$$= c_1 \frac{\cos x}{\sqrt{x}} + c_2 \frac{\cos x}{\sqrt{x}} \int x \frac{1}{x} \sec^2 x\,dx$$

$$= c_1 \frac{\cos x}{\sqrt{x}} + c_2 \frac{\cos x}{\sqrt{x}} \int \sec^2 x\,dx$$

$$= c_1 \frac{\cos x}{\sqrt{x}} + c_2 \frac{\cos x}{\sqrt{x}} \frac{\sin x}{\cos x}$$

$y(x) = c_1 \frac{\cos x}{\sqrt{x}} + c_2 \frac{\sin x}{\sqrt{x}}$ is the general solution of the given differential equation.

2-51

Solve, using the "Reduction of Order" technique: $y'' + y = \sec x$.

****

y_c SOLVES $y'' + y = 0$, SO $y_c = C_1 \cos x + C_2 \sin x$.

WE CHOOSE $y = V(x)\cos x$ ($y = V(x)\sin x$ WORKS JUST AS WELL.)

$\therefore y' = V'\cos x - V\sin x$ AND $y'' = V''\cos x - V'\sin x - V'\sin x - V\cos x$.

$\therefore y'' + y = V''\cos x - 2V'\sin x$ WHICH WE SET EQUAL TO $\sec x$.

$V''\cos x - 2V'\sin x = \sec x$. IF $W = V'$, $W' = V''$ AND WE HAVE

$W'\cos x - 2W\sin x = \sec x$ OR $W' - 2W\tan x = \sec^2 x$.

THIS IS A LINEAR EQ, WITH $P(x) = -2\tan x$ AND $Q(x) = \sec^2 x$.

$$W = e^{-\int P dx}\int e^{\int P dx} Q\, dx + C_1 e^{-\int P dx} = \frac{1}{\cos^2 x}\int \cos^2 x \sec^2 x\, dx + \frac{C_1}{\cos^2 x}$$

WE CAN SET $C_1 = 0$ IF WE DON'T, THAT TERM WILL EVENTUALLY BE ABSORBABLE INTO y_c, SO NOTHING IS GAINED BY KEEPING IT.

$\therefore W = x\sec^2 x$. $V = \int W dx$ SINCE $V' = W$, OR

$V = \int x\sec^2 x\, dx = x\tan x + \ln|\cos x| + C_2$. AGAIN, SET $C_2 = 0$ SINCE THAT TERM WILL ALSO ABSORB INTO y_c OTHERWISE.

SINCE $y = V\cos x$, WE HAVE $y = \cos x\,(x\tan x + \ln|\cos x|)$

OR $y = x\sin x + \cos x \ln|\cos x|$. THIS FUNCTION IS SUCH THAT $(D^2+1)y = \sec x$. IT IS A "PARTICULAR INTEGRAL" WHICH IS FREQUENTLY LABELLED y_p. THE GENERAL SOLUTION IS

$$y = y_c + y_p = C_1\cos x + C_2\sin x + x\sin x + \cos x \ln|\cos x|.$$

THUS OUR 2ND SOLUTION IS OF THE FORM

$$y_2 = v \cdot y_1 = k(4x-1)e^{4x} \cdot e^{-x} = k(4x-1)e^{3x}$$

GENERAL SOLUTION :

$$y = c_1 y_1 + c_2 y_2 = c_1 e^{-x} + c_2 (4x-1) e^{3x}$$

METHOD 2: FOR EQUATIONS OF THE FORM

$$a_0(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_2(x)y = 0$$

WITH 1ST SOLUTION y_1, THE 2ND SOLUTION y_2 IS GIVEN BY

$$y_2 = y_1 \int \frac{e^{-\int \frac{a_1(x)}{a_0(x)} dx}}{y_1^2} dx .$$

FOR THE EQUATION ABOVE, $y_1 = e^{-x}$, $a_0(x) = x$, AND $a_1(x) = -(2x+1)$, SO

$$\begin{aligned} y_2 &= e^{-x} \int \frac{e^{-\int \frac{-(2x+1)}{x} dx}}{(e^{-x})^2} dx \\ &= e^{-x} \int \frac{e^{\int (2 + \frac{1}{x}) dx}}{e^{-2x}} dx \\ &= e^{-x} \int e^{2x} \cdot e^{2x + \ln x} dx \\ &= e^{-x} \int x e^{4x} dx \\ &= e^{-x} \left[(\tfrac{1}{4}x - \tfrac{1}{16}) e^{4x} \right] \\ &= \tfrac{1}{16}(4x-1)e^{3x} \end{aligned}$$

2-53

Find the general solution of the differential equation

$y'' - 2y' - 8y = xe^x + 4.$

****

Find the homogeneous solution: $y'' - 2y' - 8y = 0$

characteristic equation $r^2 - 2r - 8 = 0$

$(r-4)(r+2) = 0$

$r = 4, \quad r = -2$

$y_h = c_1 e^{4x} + c_2 e^{-2x}$

Find a particular solution of the nonhomogeneous de using undetermined coefficients

guess $y_p = (Ax+B)e^x + C$

$y_p' = (Ax+B)e^x + e^x(A) = (Ax+B+A)e^x$

$y_p'' = (Ax+B+A)e^x + e^x(A) = (Ax+B+2A)e^x$

into the de: $(Ax+B+2A)e^x - 2(Ax+B+A)e^x - 8[(Ax+B)e^x + C] = xe^x + 4$

$(-9Ax - 9B)e^x - 8C = xe^x + 4$

$-9Axe^x - 9Be^x - 8C = xe^x + 4$

equating coefficients: $\begin{cases} -9A = 1 \\ -9B = 0 \\ -8C = 4 \end{cases}$

SOLUTION OF INHOMOGENEOUS EQUATIONS BY USE OF UNDETERMINED COEFFICIENTS

2-52

Find the general solution of the nonhomogeneous equation

$$y'' - 6y' + 9y = e^{3x}$$

****

The corresponding homogeneous eq. is $\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 9y = 0$ of which the characteristic equation is $r^2 - 6r + 9 = 0$

$r = 3$ is a double root of the characteristic eq. Therefore the solution Y_H of the homogeneous equation is

$$Y_H = C_1 e^{3x} + C_2 x e^{3x}$$

To find a particular solution Y_P of the given nonhomogeneous equation we take

$$Y_P = Ax^2 e^{3x}$$ (note that e^{3x}, $x e^{3x}$ are already in Y_H)

$$Y_P' = A(2x e^{3x} + 3x^2 e^{3x})$$
$$Y_P'' = A(2e^{3x} + 12x e^{3x} + 9x^2 e^{3x})$$

Substituting Y_P, Y_P', Y_P'' in the given equation

$$A(2e^{3x} + 12x e^{3x} + 9x^2 e^{3x}) - 6A(2x e^{3x} + 3x^2 e^{3x}) + 9Ax^2 e^{3x} = e^{3x}$$

Simplifying $2A e^{3x} = e^{3x}$

thus $A = \frac{1}{2}$

Therefore $Y_P = \frac{1}{2} x^2 e^{3x}$, giving

the general solution $Y(x) = Y_H + Y_P$

$$= C_1 e^{3x} + C_2 x e^{3x} + \frac{1}{2} x^2 e^{3x}$$
$$= e^{3x}(C_1 + C_2 x + \frac{1}{2} x^2)$$

2-55

Find the general solution of $y'' + y = \cos t \cos 2t$. [Hint: $\cos\alpha\cos\beta = \frac{1}{2}\cos(\beta - \alpha) + \frac{1}{2}\cos(\beta + \alpha)$.]

****

Complementary solution:

Auxiliary equation: $r^2 + 1 = (r - i)(r + i) = 0$.

Roots: $\pm i$ $(\lambda = 0, \mu = 1)$

$$y_c = C_1 e^{0t}\cos t + C_2 e^{0t}\sin t = C_1\cos t + C_2\sin t.$$

Particular solution:

$$\cos t\cos 2t = \tfrac{1}{2}\cos t + \tfrac{1}{2}\cos 3t$$

Let $g_1(t) = \frac{1}{2}\cos t$ and $g_2(t) = \frac{1}{2}\cos 3t$.

For $g_1(t)$, try $y_{p_1} = A\cos t + B\sin t$. Since these terms satisfy the homogeneous D.E., modify the initial try to:

$$y_{p_1} = t(A\cos t + B\sin t)$$
$$y'_{p_1} = (A\cos t + B\sin t) + t(-A\sin t + B\cos t)$$
$$y''_{p_1} = 2(-A\sin t + B\cos t) + t(-A\cos t - B\sin t)$$

$y''_{p_1} + y_{p_1} = g_1(t)$ leads to: $-2A = 0$, $2B = \frac{1}{2}$

$$y_{p_1} = \tfrac{1}{4}t\sin t$$

For $g_2(t) = \frac{1}{2}\cos 3t$, try $y_{p_2} = A\cos 3t + B\sin 3t$

$$y'_{p_2} = -3A\sin 3t + 3B\cos 3t$$
$$y''_{p_2} = -9A\cos 3t - 9B\sin 3t$$

$y''_{p_2} + y_{p_2} = g_2(t)$ leads to: $-8A = \frac{1}{2}$, $-8B = 0$.

$$y_{p_2} = -\tfrac{1}{16}\cos 3t$$

The general solution is:

$$y = C_1\cos t + C_2\sin t + \tfrac{1}{4}t\sin t - \tfrac{1}{16}\cos 3t.$$

so $A = -\frac{1}{9}$, $B = 0$, $C = -\frac{1}{2}$

$y_p = -\frac{1}{9} x e^x - \frac{1}{2}$

general solution: $y = c_1 e^{4x} + c_2 e^{-2x} - \frac{1}{9} x e^x - \frac{1}{2}$

2-54

Solve $y'' + y' = e^{-x}$

****

The auxiliary equation with solutions will be $m^2 + m = 0$, $m = 0$, $m = -1$.

e^{-x} corresponds to a D.E. with $m = -1$.

Consider $m = 0$, $m = -1$, $m = -1$. The solution would be $y = c_1 + c_2 e^{-x} + c_3 x e^{-x}$. We may neglect $c_1 + c_2 e^{-x}$ since it is the solution to $y'' + y' = 0$.

$$y = c_3 x e^{-x}$$
$$y' = c_3 e^{-x} - c_3 x e^{-x}$$
$$y'' = -2 c_3 e^{-x} + c_3 x e^{-x}$$

Substituting yields

$$-2 c_3 e^{-x} + c_3 x e^{-x} + c_3 e^{-x} - c_3 x e^{-x} = e^{-x}.$$

Thus $c_3 = -1$.

The general solution will be

$$y = c_1 + c_2 e^{-x} - x e^{-x}$$

2-57

Solve $y'' + 4y = 8 \sin 2x$ by the method of undetermined coefficients.

****

The general solution to this linear inhomogeneous eq. is $y = y_h + y_p$ where y_h is the general solution to the corresponding homogeneous eq. $y'' + 4y = 0$ and y_p is one particular solution (as opposed to the general solution which is what we are trying to find) to the inhomogeneous eq. $y'' + 4y = 8 \sin 2x$.

Find y_h: $y'' + 4y = 0$ Aux Eq. $r^2 + 4 = 0$

$r = \pm 2i$

$r = \alpha \pm \beta i$

$\alpha = 0, \beta = 2$

Gen. Sol.

$$y_h = c_1 e^{\alpha x}(\cos \beta x + \sin \beta x)$$

$$y_h = c_1 \cos 2x + c_2 \sin 2x.$$

Find y_p: We get the form that y_p has by differentiating $8 \sin 2x$ until we get no new kinds of functions: $8 \sin 2x$, $16 \cos 2x$, $-32 \sin 2x$, etc.

So $y_p = A \sin 2x + B \cos 2x$ and we are to find A and B so that y_p satisfies the DE.

2-56

Find the general solution to the following Euler's equation

$t^2 y'' - 2t y' + 2y = t^3$.

Hint: Use the change of variable $s = \ln t$.

****

Let $s = \ln t$, then

$$\frac{dy}{dt} = \frac{dy}{ds}\frac{ds}{dt} = \frac{1}{t}\frac{dy}{ds} \quad \text{or} \quad t\frac{dy}{dt} = \frac{dy}{ds}$$

$$\frac{d^2y}{dt^2} = -\frac{1}{t^2}\frac{dy}{ds} + \frac{1}{t^2}\frac{d^2y}{ds^2} \quad \text{or} \quad t^2\frac{d^2y}{dt^2} = \frac{d^2y}{ds^2} - \frac{dy}{ds}.$$

Substituting into the equation, we have the equivalent form

$$\frac{d^2y}{ds^2} - 3\frac{dy}{ds} + 2y = e^{3s}$$

Assume $y_c = e^{\lambda s}$, then λ satisfies the auxillary equation

$$\lambda^2 - 3\lambda + 2 = 0$$

which gives

$$\lambda = 2 \quad \text{or} \quad \lambda = 1.$$

The complementary solution

$$y_c(t) = c_1 e^{2s} + c_2 e^{s} = c_1 t^2 + c_2 t$$

where c_1 and c_2 are arbitrary constants.

We assume $y_p \equiv A e^{3s}$, A to be determined.

Since $y_p'' - 3y_p' + 2y_p = e^{3s}$, it forces

$$y_p = \frac{1}{2}e^{3s} = \frac{1}{2}t^3.$$

Hence the general solution: $y(t) = c_1 t^2 + c_2 t + \frac{1}{2}t^3$

2-58

Solve the equation: $D^3(D-1)^2y = 6+x+2e^x+\sin(x)$

****

Solution: The general solution is of the form $Y = Y_c + Y_p$ where $Y_c = c_1 + c_2x + c_3x^2 + c_4e^x + c_5xe^x$.

by undetermined coefficients

$$Y_p = Ax^3 + Bx^4 + Cx^2e^x + D\sin x + E\cos x$$

$$Y_p' = 3Ax^2 + 4Bx^3 + 2Cxe^x + Cx^2e^x + D\cos x - E\sin x$$

$$Y_p'' = 6Ax + 12Bx^2 + 2Ce^x + 2Cxe^x + 2Cxe^x + Cx^2e^x - D\sin x - E\cos x$$

$$Y_p''' = 6A + 24Bx + 6Ce^x + 4Ce^x + 6Cxe^x + 2Cxe^x + Cx^2e^x - D\cos x + E\sin x$$

$$Y_p^{(4)} = 24B + 12Ce^x + 6Ce^x + 8Cxe^x + 2Cxe^x + Cx^2e^x$$

$$Y_p^{(5)} = 20Ce^x + 8Ce^x + 10Cxe^x + 2Cxe^x + Cx^2e^x + D\sin x + E\cos x$$

$$D^3(D-1)^2Y = D^3(D^2-2D+1)Y = (D^5-2D^4+D^3)Y = Y^{(5)} - 2Y^{(4)} + Y'''$$

$$\Rightarrow 20Ce^x + 10Cxe^x + Cx^2e^x + D\cos x - E\sin x$$
$$-48B - 24Ce^x - 16Cxe^x - 2Cx^2e^x - 2D\sin x - 2E\cos x$$
$$+6A + 6Ce^x + 6Cxe^x + Cx^2e^x - D\cos x + E\sin x$$
$$+24Bx$$

$$= 6A - 48B + 24Bx + 2Ce^x - 2D\sin x - 2E\cos x = 6 + x + 2e^x + \sin x$$

$\Rightarrow E = 0$, $-2D = 1 \Rightarrow D = -1/2$, $2C = 2 \Rightarrow C = 1$

$6A - 48B = 6$, $24B = 1 \Rightarrow B = 1/24$

If we substitute y_p into $y'' + 4y = 8\sin 2x$ we get $0 = 8\sin 2x$ which is impossible. This happens because y_p has the form of the general solution to $y'' + 4y = 0$ so if we compute $y_p'' + 4y_p$ we should indeed get 0. The modification needed is to multiply y_p by x and continue (if we still get zero on the left we must multiply by x again).

$$y_p = Ax\sin 2x + Bx\cos 2x.$$

Using the product rule we get

$$y_p' = (2Ax + B)\cos 2x + (A - 2Bx)\sin 2x$$

and $$y_p'' = -4(Ax + B)\sin 2x + 4(A - Bx)\cos 2x.$$

Putting these into $y'' + 4y = 8\sin 2x$ and collecting coefficients of $\sin 2x$ and $\cos 2x$ gives

$$-4B\sin 2x + 4A\cos 2x = 8\sin 2x \qquad -4B = 8 \quad B = -2 \qquad 4A = 0 \quad A = 0$$

Equating coefficients of $\sin 2x$ on left and right and $\cos 2x$ on left and right gives $A = 2, B = 0$. So $y_p = 2x\sin 2x$. Since $y = y_h + y_p$ is the gen. sol.

$$y = C_1\cos 2x + C_2\sin 2x + 2x\sin 2x.$$

so $y_h = c_1 e^x + c_2 e^{-x} + c_3 \sin x + c_4 \cos x$

guess $y_p = Ae^{2x} + x(B\sin x + C\cos x) + D\sin 2x + E\cos 2x$

2-60

Find the general solution of the nonhomogeneous differential equation $y'' - 6y' + 5y = x^2$

****

The solution y_H of the corresponding homogeneous equation $\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 5y = 0$ is obtained by noting that the characteristic equation is

$$r^2 - 6r + 5 = 0$$
$$(r-1)(r-5) = 0$$
$$r = 1, 5$$

Therefore $y_H = c_1 e^x + c_2 e^{5x}$

To find a particular solution y_p of the given nonhomogeneous equation, we set

$$y_p = Ax^2 + Bx + C$$
$$y_p' = 2Ax + B$$
$$y_p'' = 2A$$

Substituting for y_p, y_p', y_p'' in the diff. equation

$$2A - 6(2Ax + B) + 5(Ax^2 + Bx + C) = x^2$$

that is $5Ax^2 + x(-12A + 5B) + (2A - 6B + 5C) = x^2$

Therefore $5A = 1$, $-12A + 5B = 0$, $2A - 6B + 5C = 0$

Solving $A = 1/5$, $B = 12/25$, $C = 62/125$

then $y_p = \frac{1}{5}x^2 + \frac{12}{25}x + \frac{62}{125}$

The general solution of the given equation is

$$y(x) = y_H + y_p = c_1 e^x + c_2 e^{5x} + \frac{1}{5}x^2 + \frac{12}{25}x + \frac{62}{125}$$

Then $6A - 48\frac{2}{24} = 6 \Rightarrow 6A = 8 \Rightarrow A = 4/3$

So $y_p = \frac{4}{3}x^3 + \frac{1}{24}x^4 + x^2e^x - \frac{1}{2}\sin x$

$\Rightarrow y = c_1 + c_2x + c_3x^2 + c_4e^x + c_5xe^x + y_p$

2-59

Find an appropriate guess, using the method of undetermined coefficients, for a particular solution of each nonhomogeneous differential equation.

a) $y''' - 2y'' - 3y' = x\sin x + 3x$

b) $y'''' - y = 2e^{2x} - \cos x + 3\cos 2x$

****

a) Find homogeneous solution of $y''' - 2y'' - 3y' = 0$

characteristic equation: $r^3 - 2r^2 - 3r = 0$

$r(r^2 - 2r - 3) = 0$

$r(r-3)(r+1) = 0$

$r = 0,\ r = 3,\ r = -1$

so $y_h = c_1 + c_2e^{3x} + c_3e^{-x}$

guess $y_p = (Ax+B)\sin x + (Cx+D)\cos x + x(Ex+F)$

b) find homogeneous solution of $y'''' - y = 0$

characteristic equation: $r^4 - 1 = 0$

$(r^2-1)(r^2+1) = 0$

$(r-1)(r+1)(r^2+1) = 0$

$r = 1,\ r = -1,\ r = \pm i$

2-62

Find the general solution of $L_2y = y'' - y = 2e^t - 3t^2 + \cos(2t)$

****

Characteristic equation: $r^2 - 1 = 0 \Rightarrow r = \pm 1$

$\therefore$ The solution of the homogeneous d.e. is given by $y_H = c_1e^t + c_2e^{-t}$

We now find particular solutions for each term on the right side.

(a) $L_2y = 2e^t$; Try $y_{P_1} = Ate^t$; $y'_{P_1} = Ate^t + Ae^t = Ae^t(t+1)$

$y''_{P_1} = Ae^t + A(t+1)e^t = Ae^t[t+2]$

$\therefore Ly_{P_1} = Ae^t[t+2] - Ate^t = 2e^t$

$Ate^t + 2Ae^t - Ate^t = 2e^t \Rightarrow 2Ae^t = 2e^t \Rightarrow 2A = 2$

$\therefore A = 1$, $\therefore \underline{y_{P_1} = te^t}$

(b) $L_2y = -3t^2$; Try $y_{P_2} = A + Bt + Ct^2$, $y'_{P_2} = B + 2Ct$

$y''_{P_2} = 2C$

$\therefore Ly_{P_2} = 2C - [A + Bt + Ct^2] = -3t^2$

t^0: $2C - A = 0 \Rightarrow A = 2C$

t^1: $-B = 0 \Rightarrow B = 0$

t^2: $-C = -3 \Rightarrow C = 3 \Rightarrow A = 6 \Rightarrow \underline{y_{P_2} = 6 + 3t^2}$

(c) $L_2y = \cos(2t)$; Try $y_{P_3} = A\sin(2t) + B\cos(2t)$

$y'_{P_3} = 2A\cos(2t) - 2B\sin(2t)$

$y''_{P_3} = -4A\sin(2t) - 4B\cos(2t)$

$\therefore Ly_{P_3} = -4A\sin(2t) - 4B\cos(2t) - [A\sin(2t) + B\cos(2t)] = \cos(2t)$

$\sin(2t)$: $-4A - A = 0 \Rightarrow A = 0$

$\cos(2t)$: $-4B - B = 1 \Rightarrow B = -\frac{1}{5}$ $\therefore \underline{y_{P_3} = -\frac{1}{5}\cos(2t)}$

$\therefore$ The general solution has the form $y = y_H + y_{P_1} + y_{P_2} + y_{P_3}$

or $y = c_1e^t + c_2e^{-t} + te^t + 6 + 3t^2 - \frac{1}{5}\cos(2t)$

2-61

Determine the solution of $y^{iv} - y'' - 2y = x^2 + 2$.

****

The characteristic equation is $r^4 - r^2 - 2 = 0$. This factors into $(r^2 - 2)(r^2 + 1) = 0$. Thus the characteristic values are $\pm\sqrt{2}$ and $\pm i$. This gives a solution of the homogeneous equation as $y_c = c_1 e^{\sqrt{2}x} + c_2 e^{-\sqrt{2}x} + c_3 \cos x + c_4 \sin x$.

The particular solution for the method of undetermined coefficients y_p has the form $y_p = Ax^2 + Bx + C$. Thus $y_p' = 2Ax + B$, $y_p'' = 2A$, $y_p''' = 0$ and $y_p^{iv} = 0$. Substituting these into the differential equation yields $0 - 2A - 2(Ax^2 + Bx + C) = x^2 + 2$. Therefore $-2A = 1$, $-2B = 0$ and $-2A - 2C = 2$. Therefore $A = -1/2$, $B = 0$, and $C = -1/2$. Thus $y_p = -\frac{1}{2}x^2 - \frac{1}{2}$ and the complete solution is

$$y = y_c + y_p = c_1 e^{\sqrt{2}x} + c_2 e^{-\sqrt{2}x} + c_3 \cos x + c_4 \sin x - \tfrac{1}{2}x^2 - \tfrac{1}{2}.$$

2-64

Find the general solution of $y'' + 4y' + 4y = 5\cos 3t$. Express the particular (or "steady-state") solution in the phased form $y_p = C\sin(3t - \emptyset)$.

****

The solution of the homogeneous equation is easily found to be $y_h = c_1 e^{-2t} + c_2 t e^{-2t}$.

Motivated by the right-side cosine, we try a particular solution $y_p = A\cos 3t + B\sin 3t$. The derivatives are $y_p' = -3A\sin 3t + 3B\cos 3t$ and $y_p'' = -9A\cos 3t - 9B\sin 3t$. Substituting into the equation, we get

$$-9A\cos 3t - 9B\sin 3t - 12A\sin 3t + 12B\cos 3t + 4A\cos 3t + 4B\sin 3t = 5\cos 3t + 0\sin 3t.$$

Collecting cosine and sine terms separately and equating left-side and right-side coefficients, we get two equations in A and B:

$-5A + 12B = 5$ and $-12A - 5B = 0$.

Solving simultaneously gives $A = -\frac{25}{169}$ and $B = \frac{60}{169}$.

Hence the particular solution is

$$y_p = -\frac{25}{169}\cos 3t + \frac{60}{169}\sin 3t$$

2-63

Find any particular solution for $(D^2 + 4D + 5)y = 8\sin x$

****

The roots of the auxiliary equation for the differential equation having $8\sin x$ for a particular solution are $\pm i$.
Neither of these is a root of the auxiliary equation for the given differential equation.

$\therefore y_p = A\cos x + B\sin x$

$Dy_p = -A\sin x + B\cos x$

$D^2y_p = -A\cos x - B\sin x$

Substitute into the original equation:

$$-A\cos x - B\sin x - 4A\sin x + 4B\cos x + 5A\cos x + 5B\sin x = 8\sin x$$

Equating coefficients:

$$(-A + 4B + 5A)\cos x = 0\cos x \quad \text{and} \quad (-B - 4A + 5B)\sin x = 8\sin x$$

$$\begin{aligned} 4A + 4B &= 0 \\ -4A + 4B &= 8 \\ \hline 8B &= 8 \\ B &= 1 \\ A &= -1 \end{aligned}$$

So $y_p = \sin x - \cos x$

2-65

Use the method of undetermined coefficients to find a particular solution and determine the general solution of

$$y'' + 2y' + 5y = 1 - 2e^{-x}.$$

****

Solve the homogeneous equation first.

$$y''+2y'+5y=0$$

Set $y=e^{rx} \Rightarrow r^2+2r+5=0.$

$$r = \frac{-2\pm\sqrt{2^2-4\cdot5}}{2} = -1\pm 2i$$

Complementary or homogeneous solution

$$y_c = c_1e^{-x}\cos 2x + c_2e^{-x}\sin 2x.$$

Since no part of the nonhomogeneous term occurs in the complementary solution simply assume a particular solution

$$y_p = A + Be^{-x}$$
$$y_p' = -Be^{-x}$$
$$y_p'' = Be^{-x}$$

Then $y_p''+2y_p'+5y_p = Be^{-x} - 2Be^{-x} + 5A + 5Be^{-x}$

$$= 1 - 2e^{-x}$$

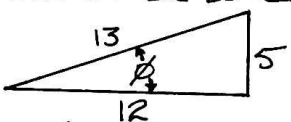
$\Rightarrow 4B = -2,\ B = -\frac{1}{2};\ 5A = 1,\ A = \frac{1}{5}$

$$y_p = \frac{1}{5} - \frac{1}{2}e^{-x}$$

Gen. Soln $\boxed{y = y_c + y_p = c_1e^{-x}\cos 2x + c_2e^{-x}\sin 2x + \frac{1}{5} - \frac{1}{2}e^{-x}}$

$$= \frac{5}{169}(12\sin 3t - 5\cos 3t).$$

In order to express this in phased form, it is convenient to construct the so-called phase triangle:



According to the triangle, $\tan\phi = 5/12$, $\cos\phi = 12/13$ and $\sin\phi = 5/13$. Hence $12 = 13\cos\phi$ and $5 = 13\sin\phi$. Using these in the particular solution, we get

$$y_p = \frac{5\cdot 13}{169}(\sin 3t\cos\phi - \cos 3t\sin\phi)$$
$$= \frac{5}{13}\sin(3t - \phi) \text{ by trig identity.}$$

Since $\tan\phi = 5/12$, $\phi = \tan^{-1}(5/12)$ $\cong 0.3948$ radians. Therefore, the general solution is $y = y_h + y_p$

$$\cong c_1e^{-2t} + c_2te^{-2t} + 0.3846\sin(3t - 0.3948).$$

Note that the period of the particular solution is $T = \frac{2\pi}{3} \cong 2.0944$.

2-67

Solve the equation $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 3y = 16xe^x$. (1)

* *

First we consider the homogeneous equation $y'' - 4y' + 3y = 0$. Assuming a solution in the form $y = e^{mx}$ gives the characteristic equation $m^2 - 4m + 3 = 0$ which has two real and distinct roots $m = 1$ and $m = 3$. Consequently, the solution of the homogeneous equation is given by

$$y_H(x) = c_1 e^x + c_2 e^{3x} \qquad (2)$$

(c_1 and c_2 being arbitrary constants.)

Next we find a particular solution $y_p(x)$ of equation (1). Note that the nonhomogeneous term on the right hand side of (1) is of the form e^{rx} with $r = 1$. Since the exponent $r = 1$ coincides with a characteristic root $m = 1$, we choose a particular solution in the form

$$y_p(x) = x(a_0 + a_1 x)e^x \qquad (3)$$

where a_0 and a_1 are undetermined constants.

From (3), it follows that

$$y_p'(x) = [a_0 + (a_0 + 2a_1)x + a_1 x^2]e^x$$

2-66

Find a general solution for:

$$y'' - 3y' + 2y = 2xe^{2x} + 3\sin 2x$$

****

We first solve the associated homogeneous equation

$$y'' - 3y' + 2y = 0$$

obtaining the complementary solution

$$y_c = c_1 e^{x} + c_2 e^{2x}.$$

Now, $(D-2)^2(D^2+4)$ annihilates $2xe^{2x} + 3\sin 2x$

Thus, a particular solution to the original equation has the form

$$y_p = Ax^2e^{2x} + Bxe^{2x} + C\cos 2x + D\sin 2x.$$

Upon differentiating, substituting and collecting like terms we have

$$2Axe^{2x} + (2A+B)e^{2x} + (-2C-6D)\cos 2x + (6C-2D)\sin 2x$$

$$= 2xe^{2x} + 3\sin 2x$$

Thus, $2A = 2$, $2A+B = 0$, $-2C-6D = 0$, $6C-2D = 3$

and we have: $A = 1$, $B = -2$, $C = 9/20$, $D = -3/20$

So the general solution is:

$$y = c_1 e^{x} + c_2 e^{2x} + x^2e^{2x} - 2xe^{2x} + (9/20)\cos 2x + (-3/20)\sin 2x$$

2-68

Determine the general solution of the differential equation $y'' - y = \cosh x$ by use of the method of undetermined coefficients.

****

The characteristic equation of the homogeneous complementary equation is given as

$$m^2 - 1 = 0 \quad \text{or} \quad m = \pm 1.$$

therefore the complementary solution is

$$y_c = c_1 e^x + c_2 e^{-x}$$

To obtain particular solution assume

$$y_p = c_1 x e^x + c_2 x e^{-x}$$

{ note: complementary solution has terms e^x, e^{-x},

substitution of y_p in the differential equation (noting that $\cosh x = \frac{e^x + e^{-x}}{2}$) yields since

$$y_p' = c_1 x e^x + c_1 e^x + c_2 e^{-x} - c_2 x e^{-x}$$

$$y_p'' = c_1 (x e^x + e^x) + c_1 e^x - c_2 e^{-x} - c_2 (-x e^{-x} + e^{-x})$$

$$c_1 x e^x + c_1 e^x + c_1 e^x - c_2 e^{-x} + c_2 x e^{-x} - c_2 e^{-x} - c_1 x e^x - c_2 x e^{-x} = \frac{e^x}{2} + \frac{e^{-x}}{2}$$

or

$$2 c_1 e^x - 2 c_2 e^{-x} = \frac{e^x}{2} + \frac{e^{-x}}{2}.$$ Therefore $c_1 = \frac{1}{4}$ $c_2 = -\frac{1}{4}$. Thus $y_p = \frac{1}{4} x e^x - \frac{1}{4} x e^{-x}$

The general solution $y = y_c + y_p$ is given by $y(x) = c_1 e^x + c_2 e^{-x} + \frac{1}{2} x \cosh x$ //

and

$$y_p'' = [\,2(a_0+a_1) + (a_0+4a_1)x + a_1x^2\,]\,e^x$$

Substituting the expressions for y_p, y_p' and y_p'' into equation (1), we have

$$[\,2(a_0+a_1) + (a_0+4a_1)x + a_1x^2\,]\,e^x$$
$$-4[\,a_0 + (a_0+2a_1)x + a_1x^2\,]\,e^x$$
$$+3[\,a_0x + a_1x^2\,]\,e^x \qquad = 16xe^x.$$

Now collecting terms with x^2e^x, xe^x and e^x respectively, we obtain

x^2e^x : $\quad a_1 - 4a_1 + 3a_1 = 0 \quad \Rightarrow \quad 0 = 0$

xe^x : $\quad (a_0+4a_1) - 4(a_0+2a_1) + 3a_0 = 16$

e^x : $\quad 2(a_0+a_1) - 4a_0 = 0$

The last two equations yield

$$a_1 = -4, \quad 2(a_1-a_0) = 0 \quad \Rightarrow \quad a_0 = -4.$$

Therefore $\qquad y_p(x) = -4x(1+x)e^x.$

Hence the general solution of equation (1) is given by

$$y(x) = y_H(x) + y_p(x)$$

i.e.

$$y(x) = c_1e^x + c_2e^{3x} - 4x(1+x)e^x.$$

2-70

Solve the differential equation

$$y'' - 3y' + 2y = 20 \sin(2x)$$

* *

Assume a solution $y = e^{mx}$ for the homogeneous equation $y'' - 3y' + 2y = 0$. Then we have the characteristic equation $m^2 - 3m + 2 = 0$ which has two real and distinct roots $m = 1$ and $m = 2$. Therefore the solution to the homogeneous equation is given by

$$y_H(x) = c_1 e^{x} + c_2 e^{2x},$$

c_1 and c_2 being arbitrary constants.

Choose a particular solution in the form

$$y_p(x) = A \cos 2x + B \sin 2x.$$

Then
$$y_p'(x) = -2A \sin 2x + 2B \cos 2x$$
$$y_p''(x) = -4A \cos 2x - 4B \sin 2x.$$

Substituting these expressions into the differential equation, we have

$$(-4A \cos 2x - 4B \sin 2x) - 3(-2A \sin 2x + 2B \cos 2x) + 2(A \cos 2x + B \sin 2x) = 20 \sin 2x.$$

Equating the coefficients of $\cos 2x$ and $\sin 2x$ respectively, we obtain

2-69

Find the general solution:

$$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} - 5y = 10\cos x - e^{-x}$$

****

THE CHARACTERISTIC EQUATION $m^2 - 4m - 5 = (m+1)(m-5) = 0$ HAS ROOTS $m_1 = -1$ AND $m_2 = 5$. THEREFORE

$$y_c = c_1 e^{-x} + c_2 e^{5x}$$

APPLYING THE METHOD OF UNDETERMINED COEFFICIENTS, WE SET

$$y_p = A\cos x + B\sin x + Cxe^{-x}$$

SO THAT

$$y_p' = -A\sin x + B\cos x + Ce^{-x} - Cxe^{-x}$$

AND

$$y_p'' = -A\cos x - B\sin x - 2Ce^{-x} + Cxe^{-x}.$$

SUBSTITUTING INTO THE ORIGINAL EQUATION AND EQUATING LIKE TERMS, WE OBTAIN

$$(-A\cos x - B\sin x - 2Ce^{-x} + Cxe^{-x}) - 4(-A\sin x + B\cos x + Ce^{-x} - Cxe^{-x}) - 5(A\cos x + B\sin x + Cxe^{-x}) = 10\cos x - e^{-x}$$

$\cos x$ TERMS: $-A - 4B - 5A = -6A - 4B = 10$

$\sin x$ TERMS: $-B + 4A - 5B = 4A - 6B = 0$

$$\Rightarrow A = \frac{-15}{13}, \quad B = \frac{-10}{13}$$

xe^{-x} TERMS: $C + 4C - 5C = 0C = 0$ ✓

e^{-x} TERMS: $-2C - 4C = -6C = -1$

$$\Rightarrow C = \frac{1}{6}$$

$$\therefore \quad y_p = \frac{-15}{13}\cos x - \frac{10}{13}\sin x + \frac{1}{6}xe^{-x}$$

GENERAL SOLUTION:

$$y = y_c + y_p = c_1 e^{-x} + c_2 e^{5x} - \frac{15}{13}\cos x - \frac{10}{13}\sin x + \frac{1}{6}xe^{-x}$$

$$y'' + y' + y = 0 \qquad (2)$$

Since the auxiliary equation of (2) $m^2 + m + 1 = 0$ has two complex roots $m_1, m_2 = -\frac{1}{2} \pm i\frac{\sqrt{3}}{2}$, the general solution of (2) is

$$y_h = e^{-\frac{x}{2}}\left(C_1 \cos\frac{\sqrt{3}}{2}x + C_2 \sin\frac{\sqrt{3}}{2}x\right)$$

ii) Find a particular solution of the inhomogeneous equation (1). Because the right hand side of (1) is a linear combination of $\cos x$ and $\sin x$, we may assume that a particular solution of (1) is of the form

$$y_p = A\cos x + B\sin x \qquad (3)$$

Plug (3) into (1), we find

$$-A\sin x + B\cos x = 2\sin x + 3\cos x \qquad (4)$$

Equating the coefficients of $\sin x$ and $\cos x$ in both sides of (4) respectively, we have

$$A = -2 \quad \text{and} \quad B = 3$$

Thus

$$y_p = -2\cos x + 3\sin x$$

iii) Now the general solution of (1) is

$$y_h + y_p = e^{-\frac{x}{2}}\left(C_1 \cos\frac{\sqrt{3}}{2}x + C_2 \sin\frac{\sqrt{3}}{2}x\right) - 2\cos x + 3\sin x$$

$\cos 2x$: $-4A - 6B + 2A = 0 \Rightarrow A = -3B$

$\sin 2x$: $-4B + 6A + 2B = 20 \Rightarrow 3A - B = 10$

Solving for A and B, we find that

$$A = 3, \quad B = -1.$$

Therefore

$$y_p(x) = 3\cos 2x - \sin 2x.$$

Thus the general solution is :

$$y(x) = y_H(x) + y_p(x)$$

i.e. $$y(x) = c_1 e^{x} + c_2 e^{2x} + 3\cos 2x - \sin 2x.$$

2-71

Find general solution of the equation

$$y'' + y' + y = 2\sin x + 3\cos x \qquad (1)$$

****

The general solution of an inhomogeneous equation is the sum of the general solution of the corresponding homogeneous equation and a particular solution of the inhomogeneous equation.

i) Find the general solution of the corresponding homogeneous equation of (1)

2-73

Solve completely the following second order linear equation:

$$y'' + 2y' + y = t\,e^{-t}$$

with the following initial conditions: $y(0) = 0$ and $y'(0) = 1$.

****

The solution $y(t)$ consists of two parts:

(i) complementary solution $y_c(t)$ which satisfies

$$y_c'' + 2y_c' + y_c = 0 \qquad (i)$$

(ii) particular solution $y_p(t)$ such that

$$y_p'' + 2y_p' + y_p = t\,e^{-t}. \qquad (ii)$$

First, assume $y_c(t) = e^{\lambda t}$ for some λ where λ is to be determined by the auxillary equation

$$\lambda^2 + 2\lambda + 1 = 0.$$

It gives $\lambda = -1$ as double roots so that

$$y_c(t) = C_1 e^{-t} + C_2 t e^{-t}$$

for some arbitrary constants C_1 and C_2. It can be checked that the above solution for $y_c(t)$ satisfies (i).

Next, since the non-homogeneous term in the equation is te^{-t} and $\lambda = -1$ is a double root, we have to choose $y_p(t)$ to be of the form

$$y_p(t) = (A + Bt)\,t^2 e^{-t}$$

where $A + Bt$ is a first degree polynomial to be determined by enforcing condition (ii).

2-72

Solve using the method of Undetermined Coefficients:

$y'' + 4y' + 4y = te^{-t}$; $y(0) = y'(0) = 0$.

****

$(D^2+4D+4)y = te^{-t}$. $(D+2)^2 y = te^{-t}$.

WE KNOW THAT $(D+1)^2 te^{-t} = 0$ BECAUSE THE SOLUTIONS TO $(D+1)^2 y = 0$ ARE e^{-t} AND te^{-t}.

$\therefore (D+1)^2(D+2)^2 y = (D+1)^2 te^{-t} = 0$ FROM WHICH

$y = c_1 e^{-t} + c_2 te^{-t} + c_3 e^{-2t} + c_4 te^{-2t}$. WE SUBSTITUTE THIS INTO THE GIVEN DIFFERENTIAL EQUATION:

$(D^2+4D+4)(c_1 e^{-t} + c_2 te^{-t} + c_3 e^{-2t} + c_4 te^{-2t}) = te^{-t}$ BUT SINCE

$(D^2+4D+4)(c_3 e^{-2t} + c_4 te^{-2t}) = 0$, SINCE $c_3 e^{-2t} + c_4 te^{-2t}$ IS THE SOLUTION TO THE REDUCED EQUATION $(D^2+4D+4)y = 0$,

$(D^2+4D+4)(c_1 e^{-t} + c_2 te^{-t}) = te^{-t}$ "DETERMINES" c_1 AND c_2.

BY DOING THE DIFFERENTIATIONS, $c_1 e^{-t} + 2c_2 e^{-t} + c_2 te^{-t} = te^{-t}$,

AND $c_2 = 1$. SINCE $c_1 + 2c_2 = 0$, $c_1 = -2$.

$\therefore y = c_3 e^{-2t} + c_4 te^{-2t} - 2e^{-t} + te^{-t}$.

$y(0) = 0$ YIELDS $c_3 - 2 = 0$ OR $c_3 = 2$.

$\therefore y = 2e^{-2t} + c_4 te^{-2t} - 2e^{-t} + te^{-t}$. DIFFERENTIATING,

$y' = -4e^{-2t} + c_4 e^{-2t} - 2c_4 te^{-2t} + 2e^{-t} + e^{-t} - te^{-t}$

$y'(0) = 0$ YIELDS $-4 + c_4 + 3 = 0$ OR $c_4 = 1$. THEN

$y(t) = -2e^{-t} + te^{-t} + 2e^{-2t} + te^{-2t}$.

The method of undetermined coefficients says that y_p should be in the span of $\{1, x, \cos x, \sin x\}$. None of these are solutions to the homogeneous, so we don't need the modification rule.

$$y_p = ax + b + c\cos x + d\sin x$$
$$y_p' = a + d\cos x - c\sin x$$
$$y_p'' = -c\cos x - d\sin x$$
$$2y_p' = 2a + 2d\cos x - 2c\sin x$$
$$2y_p = 2ax + 2b + 2c\cos x + 2d\sin x$$

$$(2a)x + (2a+2b) + (c+2d)\cos x + (-2c+d)\sin x$$
$$= 1 \qquad = 0 \qquad = 0 \qquad = 1$$

$$2a = 1 \Rightarrow a = 1/2$$
$$2a+2b = 0 \Rightarrow b = -1/2$$

$$c + 2d = 0 \qquad 2c + 4d = 0$$
$$-2c + d = 1 \qquad -2c + d = 1$$
$$\overline{\qquad\qquad 5d = 1}$$

$$d = 1/5, \quad c = -2/5$$

$$y_p'(t) = (3At^2 + 2Bt)e^{-t} + (-t^2)(At+B)e^{-t}$$

$$y_p''(t) = (6At + 2B)e^{-t} - 2(3At^2 + 2Bt)e^{-t} + t^2(At+B)e^{-t}$$

$$y_p''(t) + 2y_p'(t) + y_p(t) = (6At + 2B)e^{-t} \equiv te^{-t}$$

which forces $A = \frac{1}{6}$ and $B = 0$.

The complete general solution is then

$$y(t) = c_1 e^{-t} + c_2 te^{-t} + \tfrac{1}{6}t^3 e^{-t}$$

and

$$y'(t) = -c_1 e^{-t} + c_2(e^{-t} - te^{-t}) + \tfrac{1}{2}t^2 e^{-t} - \tfrac{1}{6}t^3 e^{-t}.$$

Use the initial conditions: $y(0) = 0$ and $y'(0) = 1$ to determine c_1 and c_2.

$$y(0) = 0 \Rightarrow c_1 = 0$$

$$y'(0) = 1 \Rightarrow c_2 = 1$$

Hence the final solution is given by

$$y(t) = te^{-t} + \tfrac{1}{6}t^3 e^{-t}.$$

2-74

Solve for y if $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 2y = x + \sin x$.

****

First solve the homogeneous:

$$\lambda^2 + 2\lambda + 2 = 0, \quad \lambda = -1 \pm i$$

$$\boxed{Y_h = e^{-x}[A\cos x + B\sin x]}$$

2-76

$y''' - 3y'' + 3y' - y = 0.$ $y =$ (A) Ce^x

(B) $C_1e^x + C_2e^{2x} + C_3e^{3x}$

(C) Cx^2e^x

(D) $C_2x^2e^x + C_1xe^x + C_0e^x$

(E) $C_2x^2e^x + C_1xe^x + C_0e^x + C.$

****

We presume that y is some exponential function and substitute $y = e^{mx}$ where m is some real or complex number then

$$y' = me^{mx} \qquad y'' = m^2e^{mx} \qquad y''' = m^3e^{mx}$$

Substituting y, y', y'', y''' into the equation we obtain

$$e^{mx}(m^3 - 3m^2 + 3m - 1) = 0$$

Since e^{mx} is not identically zero we must have

$$m^3 - 3m^2 + 3m - 1 = 0$$

$$(m-1)^3 = 0.$$

$$m = 1.$$

Thus, Ce^x is a solution. However, the solution of a third order equation must be the linear combination of three linearly independent functions. If we had three different m's, m_1, m_2, m_3, it would be $C_1e^{m_1x} + C_2e^{m_2x} + C_3e^{m_3x}$. Having $m_1 = m_2 = m_3 = 1$, we assume that the solution is

Thus

$$y = e^{-x}[A\cos x + B\sin x] + \frac{1}{2}x - \frac{1}{2} - \frac{2}{5}\cos x + \frac{1}{5}\sin x$$

2-75

Find the form of a particular integral for each of the following non-homogeneous differential equations. Do not evaluate the constants.

(a) $y'' + 2y' + y = (t^2 + 3)\, e^{-t} + e^{-t}\cos t + t\, e^{2t}$

(b) $y'' + 2\, y' + 2y = t^3\, e^{-t} + t\, e^{-t}\sin t + (t^2 + 3)\, e^{-t}\cos 2t.$

****

(a) The auxillary equation is $\lambda^2+2\lambda+1=0$ giving $\lambda=-1$ as double roots.

$$y_p(t) = (A_0t^2 + A_1t + A_2)\,t^2e^{-t} + B_0e^{-t}\cos t + C_0e^{-t}\sin t + (D_0t + D_1)\,e^{2t}$$

Notice that $y_p(t)$ contains the term $(A_0t^2+A_1t+A_2)t^2e^{-t}$ since $\lambda=-1$ is a repeated root of the auxillary equation.

(b) The auxillary equation is $\lambda^2+2\lambda+2=0$ giving $\lambda=-1\pm i$.

$$y_p(t) = (A_0t^3 + A_1t^2 + A_2t + A_3)e^{-t} + (B_0t+B_1)\,t\,e^{-t}\sin t + (C_0t + C_1)\,t\,e^{-t}\sin t + (D_0t^2 + D_1t + D_2)\,e^{-t}\cos 2t + (E_0t^2 + E_1t + E_2)\,e^{-t}\sin 2t$$

2-77

Solve the differential equation: $y'' + y = 5x$, $y(0) = -1$, $y'(0) = 6$.

****

$y'' + y = 5x$

Solve the associated homogeneous differential equation $y'' + y = 0$

$m^2 + 1 = 0 \Rightarrow m = \pm i$

$y_c = c_1 \sin x + c_2 \cos x$

The form of the particular solution is

$y_p = Ax + B$

$y_p' = A$

$y_p'' = 0$

So $y_p'' + y_p = 0 + Ax + B = 5x$

$\Rightarrow A = 5, \; B = 0$

$y = y_c + y_p = c_1 \sin x + c_2 \cos x + 5x$

$y(0) = -1 = c_1 \cdot 0 + c_2 \cdot 1 + 5 \cdot 0 \Rightarrow c_2 = -1$

$y = c_1 \sin x - \cos x + 5x$

$y' = c_1 \cos x + \sin x + 5$

so $y'(0) = 6 = c_1 \cos 0 + 0 + 5 \Rightarrow 6 = c_1 + 5$

$\&$ $c_1 = 1$

$y = \sin x - \cos x + 5x$

$$y(x) = C(x)e^x$$

where the parameter $C(x)$ is a function of x and varies with x.

Using the product rule to differentiate, we obtain

$$y'(x) = C'e^x + Ce^x$$

$$y''(x) = C''e^x + 2C'e^x + Ce^x$$

$$y'''(x) = C'''e^x + 3C''e^x + 3C'e^x + Ce^x.$$

Substituting into the equation,

$$(C'''e^x + 3C''e^x + 3C'e^x + Ce^x) - 3(C''e^x + 2C'e^x + Ce^x) + 3(C'e^x + Ce^x) - Ce^x = 0.$$

$$e^x[C''' + 3C'' + 3C' + C - 3C'' - 6C' - 3C + 3C' + 3C - C] = 0$$

$$e^x C'''(x) = 0.$$

Since e^x is not identically zero, we must have

$$C'''(x) = 0$$

Integrating,

$$C''(x) = a \quad \text{(a constant)}$$

$$C'(x) = ax + b \quad \text{(a second constant)}$$

$$C(x) = \tfrac{1}{2}ax^2 + bx + c \quad \text{(a third constant)}$$

Hence, the solution is

$$y(x) = C(x)e^x = \tfrac{1}{2}ax^2e^x + bxe^x + ce^x$$

That is a linear combination of the three linearly independent functions x^2e^x, xe^x and e^x.

The correct answer is (D).

THE METHOD OF VARIATION OF PARAMETERS

2-79

$y_1(x) = x$ and $y_2(x) = e^x$ are linearly independent solutions of the homogeneous equation $(x-1)y'' - xy' + y = 0$. Find a particular solution of $(x-1)y'' - xy' + y = (x-1)^2 e^{2x}$.

****

First write the equation as

$$y'' - \frac{x}{x-1}y + \frac{1}{x-1}y = (x-1)e^{2x} = g(x).$$

If $y_p(x) = V_1(x)y_1(x) + V_2(x)y_2(x)$, then

$$V_1(x) = \int \frac{-g(x)y_2(x)}{W[y_1,y_2](x)}\,dx,$$

$$V_2(x) = \int \frac{g(x)y_1(x)}{W[y_1,y_2](x)}\,dx,$$

where $W[y_1,y_2](x) = y_1(x)y_2'(x) - y_1'(x)y_2(x)$.

$$W[y_1,y_2](x) = xe^x - 1\cdot e^x = (x-1)e^x$$

$$V_1(x) = \int \frac{-(x-1)e^{2x}\cdot e^x}{(x-1)e^x}\,dx = -\frac{1}{2}e^{2x} + C_1$$

$$V_2(x) = \int \frac{(x-1)e^{2x}\cdot x}{(x-1)e^x}\,dx = \int xe^x\,dx$$

$$= xe^x - e^x + C_2 = (x-1)e^x + C_2$$

Take $C_1 = C_2 = 0$ to obtain the particular solution

$$y_p(x) = -\frac{1}{2}xe^{2x} + (x-1)e^{2x}$$

$$= \left(\frac{1}{2}x - 1\right)e^{2x}.$$

2-78

Find the general solution of $y''+9y = -6xe^{3x}$.

****

The characteristic equation is $0 = r^2+9$, so $r = \pm 3j$ and the general solution of the homogeneous equation $y''+9y=0$ is $y_c(x) = c_1 \cos 3x + c_2 \sin 3x$. Use undetermined coefficients to find a particular solution of the form

$y_p(x) = (Ax+B)e^{3x}$

$y_p'(x) = Ae^{3x} + (Ax+B)3e^{3x} = 3Axe^{3x} + (A+3B)e^{3x}$

$y_p''(x) = 9Axe^{3x} + 3Ae^{3x} + (3A+9B)e^{3x} = 9Axe^{3x} + (6A+9B)e^{3x}$

Then

$y_p'' + 9y_p = 9Axe^{3x} + (6A+9B)e^{3x} + 9Axe^{3x} + 9Be^{3x} = -6xe^{3x}$

$18Axe^{3x} + (6A+18B)e^{3x} = -6xe^{3x} + 0e^{3x}$

Then $18A = -6$; $A = -\frac{1}{3}$

$6A+18B = 0$; $-2+18B = 0$; $B = \frac{1}{9}$

Therefore $y_p = (-\frac{1}{3}x + \frac{1}{9})e^{3x}$ and the general solution of $y''+9y = -6xe^{3x}$ is

$y(x) = y_c + y_p = c_1 \cos 3x + c_2 \sin 3x + (-\frac{1}{3}x + \frac{1}{9})e^{3x}$

2-81

Check that x+1 and e^x are solutions of

$$xy'' - (x+1)y' + y = 0 \qquad (1)$$

find the general solution of

$$xy'' - (x+1)y' + y = -x^2, \quad x > 0 \qquad (2)$$

****

The general solution of (1) is given by $A(x+1) + Be^x$ (A, B arbitrary constants).

Find a particular solution of (2) of the form

$$y_p(x) = (x+1)A(x) + e^x B(x).$$

Substituting $y_p(x)$ into (2) and setting $(x+1)A' + e^x B' = 0$ we have

$$\begin{cases} (x+1)A' + e^x B' = 0 \\ A' + e^x B' = -x \end{cases}$$

$$A' = 1, \quad A = x$$

$$e^x B' = -(x+1), \quad B' = -(x+1)e^{-x}$$

$$B(x) = (x+2)e^{-x}$$

so $y_p(x) = x(x+1) + (x+2) = x^2 + 2x + 2$ or $y_p(x) = x^2$.

Therefore the general solution of (2) is given by

$$y(x) = C_1(x+1) + C_2 e^x + x^2.$$

2-80

Find the general solution of the equation

$$(x^2-1)y'' - 2xy' + 2y = (x^2-1)^2 \qquad (1)$$

****

Step 1. Find general solution of

$$(x^2-1)y'' - 2xy' + 2y = 0 \qquad (2)$$

By inspection, $y_1 = x$ is a solution of (2). Assume $y_2 = vx$, then $v = \int \frac{1}{x^2} e^{\int \frac{2x}{x^2-1}dx} dx = x + \frac{1}{x}$, Hence the general solution of (2) is $C_1 x + C_2(x^2+1)$.

Step 2. Use variation of parameters to find a particular solution of (1). Write (1) as

$$y'' - \frac{2x}{x^2-1}y' + \frac{2}{x^2-1}y = x^2-1$$

Assume $y_p = v_1 x + v_2(x^2+1)$. Applying the formula

$$v_1 = \int \frac{-y_2 R(x)}{W(y_1, y_2)} dx \qquad v_2 = \int \frac{y_1 R(x)}{W(y_1, y_2)} dx$$

where $R(x) = x^2-1$, $W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = x^2-1$

we get $v_1 = -\frac{x^3}{3} - x \qquad v_2 = \frac{1}{2}x^2$

Hence $y_p = \frac{1}{6}x^4 - \frac{1}{2}x^2$

The general solution of (1) is

$$C_1 y_1 + C_2 y_2 + y_p = C_1 x + C_2(x^2+1) + \frac{1}{6}x^4 - \frac{1}{2}x^2$$

2-83

Find a particular solution of

$$y''(x) + y(x) = \operatorname{cosec} x$$

by the method of variation of parameters.

****

The corresponding homogeneous solution is $y'' + y = 0$. Therefore the complementary solution is $y_c(x) = c_1 \sin x + c_2 \cos x$.

Write $y_1(x) = \sin x$ and $y_2(x) = \cos x$; the Wronskian

$$W(y_1, y_2; x) = \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix} = \begin{vmatrix} \sin x & \cos x \\ \cos x & -\sin x \end{vmatrix} = -1.$$

The particular solution $y_p(x)$ is given by

$$y_p(x) = V_1(x)\, y_1(x) + V_2(x)\, y_2(x)$$

where

$$V_1(x) = \int \frac{-y_2(x) \operatorname{cosec} x}{W(y_1, y_2)}\, dx = \int \frac{-\cos x \operatorname{cosec} x}{-1}\, dx$$

$$= \int \frac{\cos x}{\sin x}\, dx = \ln |\sin x|$$

and

$$V_2(x) = \int \frac{y_1(x) \operatorname{cosec} x}{W(y_1, y_2)}\, dx = \int \frac{\sin x \operatorname{cosec} x}{-1}\, dx = -x.$$

Hence

$$y_p(x) = \sin x \ln |\sin x| - x \cos x$$

and the overall solution

$$y(x) = c_1 \sin x + c_2 \cos x + \sin x \ln |\sin x| - x \cos x.$$

used

2-82

Solve the differential equation

$$y'' + y = \csc t \cot t$$

and write the solution in real form.

****

Homogeneous d.e.: $y'' + y = 0$

$r^2 + 1 = 0 \Rightarrow r = \pm i \Rightarrow e^{it}, e^{-it}$ are complex solutions; $\cos t$ and $\sin t$ are the corresponding real solutions. $\therefore y_H = a_1 \underbrace{\cos t}_{y_1} + a_2 \underbrace{\sin t}_{y_2}$

To find a particular solution, try $y_p = u_1(t) y_1 + u_2(t) y_2$

then $\textcircled{*}\begin{cases} u_1' \cos t + u_2' \sin t = 0 \\ u_1'(-\sin t) + u_2' \cos t = \csc t \cot t = \dfrac{\cos t}{\sin^2 t} \end{cases}$

Solving $\textcircled{*}$ for u_1' and u_2': (Using Cramer's Rule here - but any method for solving is O.K.)

$$u_1'(t) = \frac{\begin{vmatrix} 0 & \sin t \\ \frac{\cos t}{\sin^2 t} & \cos t \end{vmatrix}}{1 = \begin{vmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{vmatrix}} = -\frac{\cos t}{\sin t}; \quad u_2'(t) = \frac{\begin{vmatrix} \cos t & 0 \\ -\sin t & \frac{\cos t}{\sin^2 t} \end{vmatrix}}{\begin{vmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{vmatrix}} = \frac{\cos^2 t}{\sin^2 t}$$

$$u_1'(t) = -\frac{\cos t}{\sin t} \qquad \text{or} \quad u_2'(t) = \cot^2 t = \csc^2 t - 1$$

$$\Rightarrow u_1(t) = -\int \frac{\cos t}{\sin t} = -\ln|\sin t| \qquad \Rightarrow u_2(t) = \int \csc^2 t\, dt - \int dt = -\cot(t) - t$$

$$\therefore y_p = \cos t(-\ln|\sin t|) + \sin t(-\cot t - t) = -\cos t \ln|\sin t| - \cos t - t \sin t$$

The general solution has the form $y = a_1 y_1 + a_2 y_2 + y_p$

$$\therefore y = a_1 \cos t + a_2 \sin t - \cos t \ln|\sin t| - \cos t - t \sin t$$

or without any loss of generality

$$y = a_1 \cos t + a_2 \sin t - \cos t \ln|\sin t| - t \sin t$$

2-85

Solve x y" + (2+x) y' + y = e^{-x}, with y_1 = 1/x being a solution of the homogeneous equation.

****

$y'' + \frac{2+x}{x} y' + \frac{1}{x} y = 0$ has one solution $y_1 = \frac{1}{x}$

Find second solution by reduction of order

$y_2 = u \cdot \frac{1}{x}$ $\quad y_2' = u' \cdot \frac{1}{x} - \frac{1}{x^2} u$

$y_2'' = u'' \cdot \frac{1}{x} - \frac{2}{x^2} u' + \frac{2}{x^3} u$

into the de: $\left(\frac{1}{x} u'' - \frac{2}{x^2} u' + \frac{2}{x^3} u\right) + \left(\frac{2}{x} + 1\right)\left(\frac{1}{x} u' - \frac{1}{x^2} u\right) + \left(\frac{1}{x}\right)\left(\frac{1}{x} u\right) = 0$

$\frac{1}{x} u'' + \frac{1}{x} u' = 0$

$u'' + u' = 0$ $\quad$ let $w = u'$ $\quad w' = u''$

$w' + w = 0$

$\frac{dw}{dx} + w = 0$

separating variables: $\frac{dw}{w} = -dx$

$\ln w = -x$

$w = e^{-x}$

so $u' = e^{-x}$ $\quad u = -e^{-x}$

and $y_2 = \frac{-e^{-x}}{x}$

homogeneous solution: $y_h = c_1 \frac{1}{x} + c_2 \frac{e^{-x}}{x}$

find a particular solution of the nonhomogeneous de using variation of parameters

2-84

Find a general solution for:

$$y'' + 4y = \sec 2x \tan 2x$$

****

We first solve the associated homogeneous equation

$$y'' + 4y = 0$$

obtaining the complementary solution

$$y_c = c_1 \cos 2x + c_2 \sin 2x$$

We must apply the method of Variation of Parameters to determine a particular solution.

Let $y_p = u_1 \cos 2x + u_2 \sin 2x$.

Then $$u_1 = \int \frac{(-\sec 2x \tan 2x) \cdot (\sin 2x)}{W(\cos 2x, \sin 2x)}\, dx$$

$$= \int \frac{-\tan^2 2x}{2}\, dx = \frac{x}{2} - \frac{1}{4}\tan 2x$$

and $$u_2 = \int \frac{(\sec 2x \tan 2x) \cdot (\cos 2x)}{W(\cos 2x, \sin 2x)}\, dx$$

$$= \int \frac{\tan 2x}{2}\, dx = \frac{1}{4}\ln|\sec 2x|$$

Thus $$y_p = \left(\frac{x}{2} - \frac{1}{4}\tan 2x\right)\cos 2x + \left(\frac{1}{4}\ln|\sec 2x|\right)\sin 2x$$

and

$$y = c_1 \cos 2x + c_2 \sin 2x + \frac{x}{2}\cos 2x + \frac{\sin 2x}{4}\ln|\sec 2x|$$

(Note: $-\frac{1}{4}\tan 2x \cos 2x = -\frac{1}{4}\sin 2x$ is contained in $c_2 \sin 2x$.)

2-86

Solve $x^2y'' + 4xy' + 2y = e^{-x}$.

****

Since the homogeneous is an Euler equation we must use variation of parameters. First solve the homogeneous:

$$x^2y'' + 4xy' + 2y = 0 \quad , \text{ guess } y = x^m$$

$$\Rightarrow \quad m(m-1) + 4m + 2 = 0$$

$$m^2 + 3m + 2 = 0, \quad m = -1 \text{ or } -2$$

$$\boxed{y_h = \frac{a}{x} + \frac{b}{x^2}}$$

Now try

$$y_p = \frac{A}{x} + \frac{B}{x^2}$$

$$y_p' = \frac{-A}{x^2} - \frac{2B}{x^3} + \boxed{\frac{A'}{x} + \frac{B'}{x^2}} \equiv 0$$

$$y_p'' = \frac{2A}{x^3} + \frac{6B}{x^4} + \frac{-A'}{x^2} + \frac{-2B'}{x^3}$$

$$x^2y_p'' = \frac{2A}{x} + \frac{6B}{x^2} + -A' + \frac{-2B'}{x}$$

$$4xy_p' = \frac{-4A}{x} - \frac{8B}{x^2}$$

$$2y_p' = \frac{2A}{x} + \frac{2B}{x^2}$$

seek $y_p = u_1(x)\, x^{-1} + u_2(x)\, e^{-x} x^{-1}$

solve simultaneously: $\begin{cases} y_1 u_1' + y_2 u_2' = 0 \\ y_1' u_1' + y_2' u_2' = f(x) \end{cases}$

where $y_1 = x^{-1}$, $y_2 = e^{-x}x^{-1}$, $f(x) = e^{-x}x^{-1}$

(note: $f(x)$ is the right hand side of the de when the lead coefficient is one)

solve: $\begin{cases} x^{-1} u_1' + e^{-x} x^{-1} u_2' = 0 \\ -x^{-2} u_1' + (-e^{-x}x^{-2} - e^{-x}x^{-1})\, u_2' = e^{-x}x^{-1} \end{cases}$

$\begin{cases} x^{-1} u_1' + e^{-x} x^{-1} u_2' = 0 \\ -x^{-1} u_1' + (-e^{-x}x^{-1} - e^{-x})\, u_2' = e^{-x} \end{cases}$ (multiplying 2nd eqn by x)

(adding together)

$$-e^{-x} u_2' = e^{-x}$$

$$u_2' = -1 \qquad \text{so } u_2 = -x$$

$$x^{-1} u_1' + e^{-x} x^{-1}(-1) = 0$$

$$u_1' = e^{-x} \qquad \text{so } u_1 = -e^{-x}$$

thus $y_p = -e^{-x}x^{-1} - xe^{-x}x^{-1} = -\frac{e^{-x}}{x} - e^{-x}$

general solution: $y = c_1 \frac{1}{x} + \underbrace{c_2 \frac{e^{-x}}{x} - \frac{e^{-x}}{x}}_{\text{combine with a new constant}} - e^{-x}$

$$= c_1 \frac{1}{x} + c_3 \frac{e^{-x}}{x} - e^{-x}$$

2-87

Find the general solution of

$$ty''(t) + (1 - 2t)\, y'(t) + (t-1)\, y(t) = t\, e^t, \quad t > 0,$$

knowing that one solution of the corresponding problem is $y_1(t) = e^t$.

****

Assume $y(t) = v(t)\, y_1(t) = e^t v(t)$, then its derivatives are

$$y'(t) = v(t) e^t + v'(t) e^t$$

$$y''(t) = v(t) e^t + 2 v'(t) e^t + v''(t) e^t.$$

Substitute into the equation we obtain

$$te^t v''(t) + e^t v'(t) = te^t$$

which is reduced to

$$t v''(t) + v'(t) = t.$$

The two terms on the left hand side can be written as

$$\frac{d}{dt}\left(t \frac{dv}{dt}\right) = t$$

$$\frac{dv}{dt} = \frac{1}{2}t + \frac{c_2}{t}$$

one further integration

$$v(t) = \frac{t^2}{4} + c_2 \ln t + c_1.$$

The general solution

$$y(t) = v(t) e^t = c_1 e^t + c_2 e^t \ln t + \frac{1}{4} t^2 e^t.$$

Notice that the second linearly independent solution

$$y_2(t) = e^t \ln t$$

and the particular solution

$$y_p(t) = \frac{1}{4} t^2 e^t.$$

Plugging these into the equation, all the A & B stuff cancels, leaving:

$$-A' - \frac{2B'}{x} = e^{-x}$$

and $\frac{A'}{x} + \frac{B'}{x^2} = 0$ or $A' + \frac{B'}{x} = 0$

Adding we get $-\frac{B'}{x} = e^{-x}$

so $B' = -xe^{-x}$; $A' = e^{-x}$

$$\Rightarrow \quad A = -e^{-x} + a$$

$$B = xe^{-x} + e^{-x} + b$$

$$y = \frac{(-e^{-x} + a)}{x} + \frac{xe^{-x} + e^{-x} + b}{x^2}$$

or

$$\boxed{y = \frac{a}{x} + \frac{b}{x^2} + \frac{e^{-x}}{x^2}}$$

2-89

Given that $y_h = c_1e^{-x} + c_2e^{-x}/x$ find a particular solution to

$$xy'' + 2(x + 1)y' + (x + 2)y = e^{-x}/x, \quad x > 0$$

Using variation of parameters.

****

Here it is given that $y_1 = e^{-x}$, $y_2 = e^{-x}/x$

Then $y_p = e^{-x}u_1 + \frac{e^{-x}}{x}u_2$

It is known that u_1 and u_2 satisfy

$$u_1'y_1 + u_2'y_2 = 0$$
$$u_1'y_1' + u_2'y_2' = f(x),$$

when y_p satisfies $y'' + p(x)y' + q(x)y = f(x)$.

The nonhomogeneous term $f(x) = e^{-x}/x^2$.

Solve then $u_1'e^{-x} + u_2'e^{-x}/x = 0$

$$-u_1'e^{-x} + u_2'\left(-\frac{e^{-x}}{x} - \frac{e^{-x}}{x^2}\right) = \frac{e^{-x}}{x^2}$$

Since e^{-x} is a common factor,

$$u_1' + u_2'/x = 0 \Rightarrow u_2' = -xu_1'$$
$$-u_1' + u_2'\left(-\frac{1}{x} - \frac{1}{x^2}\right) = \frac{1}{x^2}$$

Eliminate u_2': $-u_1' - xu_1'\left(-\frac{1}{x} - \frac{1}{x^2}\right) = \frac{1}{x^2}$

$$\Rightarrow u_1' = \frac{1}{x} \Rightarrow u_1 = \ln x, \text{ since } x > 0$$

$u_2' = -1 \Rightarrow u_2 = -x$

$$\boxed{y_p = e^{-x}\ln x - e^{-x}}$$

This is part of the homogeneous solution

2-88

Use the method of variation of parameters to solve the differential equation $y'' + y' = e^x$.

****

The characteristic equation is $r^2+r=0$. The roots are $r=0$ and $r=-1$. Thus the complementary solution is

$$y_c = c_1 + c_2 e^{-x}.$$

Our trial solution for the method of variation of parameters is

$$y_p = u_1(x) + u_2(x)\, e^{-x}.$$

Differentiate this to obtain

$$y_p' = u_1' + u_2(-e^{-x}) + e^{-x}u_2'.$$

Let $u_1' + e^{-x}u_2' = 0$.

This gives $y_p' = -u_2 e^{-x}$.

Differentiate again to obtain

$$y_p'' = u_2 e^{-x} - e^{-x}u_2'.$$

Substitute into the differential equation to obtain

$$y_p'' + y_p' = u_2 e^{-x} - e^{-x}u_2' - u_2 e^{-x} = e^x.$$

This gives $-e^{-x}u_2' = e^x$ or

$$u_2' = -e^{2x}.$$

Integrate this to obtain $u_2 = -\frac{1}{2}e^{2x}$.

Since $u_1' + e^{-x}u_2' = 0$, we also have

$$u_1' = -e^{-x}u_2' = -e^{-x}(-e^{2x}) = e^x.$$

Thus after integration $u_1 = e^x$. This gives $y_p = e^x + \left(-\frac{1}{2}e^{2x}\right)e^{-x} = \frac{1}{2}e^x$.

Therefore $y = y_c + y_p = c_1 + c_2 e^{-x} + \frac{1}{2}e^x$.

2-91

Solve the following equation using the method of variation of parameters.

$$y'' - 2y' + y = 6x^4 e^x.$$

* *

For the homogeneous equation $y'' - 2y' + y = 0$, assume a solution of the form $y = e^{mx}$. This leads to the characteristic equation $m^2 - 2m + 1 = 0$, i.e. $(m-1)^2 = 0$. Since we have a repeated root $m = 1$, the two linearly independent solutions of the homogeneous equation are

$$y_1(x) = e^x, \quad y_2 = xe^x.$$

By the method of variation of parameters, a particular solution of equation (1) is given by

$$y_p(x) = u_1(x) y_1(x) + u_2(x) y_2(x)$$

where

$$u_1(x) = -\int \frac{f(x) y_2(x)}{W(y_1, y_2)} dx,$$

$$u_2(x) = \int \frac{f(x) y_1(x)}{W(y_1, y_2)} dx.$$

Here

$$f(x) = 6x^4 e^x$$

2-90

Find the general solution of $y''+9y = 2\sec 3x$.

****

The general solution of $y''+9y = 0$ is $y_c = c_1\cos 3x + c_2\sin 3x$, so the variation of parameters solution has the form

$y_p = u_1(x)\cos 3x + u_2(x)\sin 3x$

$y_p' = -3u_1\sin 3x + 3u_2\cos 3x$ if $u_1'\cos 3x + u_2'\sin 3x = 0$

$y_p'' = -3u_1'\sin 3x - 9u_1\cos 3x + 3u_2'\cos 3x - 9u_2\sin 3x$

Then

$$y_p'' + 9y_p = -3u_1'\sin 3x - 9u_1\cos 3x + 3u_2'\cos 3x - 9u_2\sin 3x + 9u_1\cos 3x + 9u_2\sin 3x$$

$$-3u_1'\sin 3x + 3u_2'\cos 3x = 2\sec 3x$$

Solve the system $\begin{cases} u_1'\cos 3x + u_2'\sin 3x = 0 \\ -3u_1'\sin 3x + 3u_2'\cos 3x = 2\sec 3x \end{cases}$

By Cramer's Rule,

$$u_1' = \frac{\begin{vmatrix} 0 & \sin 3x \\ 2\sec 3x & 3\cos 3x \end{vmatrix}}{\begin{vmatrix} \cos 3x & \sin 3x \\ -3\sin 3x & 3\cos 3x \end{vmatrix}} = \frac{-2\sec 3x\sin 3x}{3}$$

and

$$u_2' = \frac{\begin{vmatrix} \cos 3x & 0 \\ -3\sin 3x & 2\sec 3x \end{vmatrix}}{3} = \frac{2\cos 3x\sec 3x}{3} = \frac{2}{3}$$

Then $u_1 = \int -\frac{2}{3}\tan 3x\,dx = \frac{2}{9}\ln|\cos 3x|$

$u_2 = \int \frac{2}{3}\,dx = \frac{2}{3}x$

Therefore, $y_p = u_1\cos 3x + u_2\sin 3x = \frac{2}{9}\ln|\cos 3x|\cos 3x + \frac{2}{3}x\sin 3x$

the general solution is

$$y(x) = c_1\cos 3x + c_2\sin 3x + \frac{2}{9}\ln|\cos 3x|\cos 3x + \frac{2}{3}x\sin 3x$$

2-92

Solve the differential equation: $y'' - 2y' + y = \frac{e^x}{x}$

****

find the complementary solution first:

$y'' - 2y' + y = 0$

$m^2 - 2m + 1 = 0 \Rightarrow (m-1)^2 = 0 \Rightarrow m = +1, +1$

so $y_c = c_1 e^x + c_2 x e^x$

The method of variation of parameters must be used since $\frac{e^x}{x}$ is not a sum or product of e^x or x^n, $n > 0$.

let $y = u_1 e^x + u_2 x e^x$

$$u_1' = \frac{\begin{vmatrix} 0 & xe^x \\ \frac{e^x}{x} & xe^x + e^x \end{vmatrix}}{\begin{vmatrix} e^x & xe^x \\ e^x & xe^x + e^x \end{vmatrix}} = \frac{0 - xe^x \cdot \frac{e^x}{x}}{xe^{2x} + e^{2x} - xe^{2x}} = \frac{-e^{+2x}}{e^{2x}} = -1$$

so $u_1 = \int -1\, dx = -x + c_1$

$$u_2' = \frac{\begin{vmatrix} e^x & 0 \\ e^x & \frac{e^x}{x} \end{vmatrix}}{\begin{vmatrix} e^x & xe^x \\ e^x & xe^x + e^x \end{vmatrix}} = \frac{\frac{e^{2x}}{x} - 0}{e^{2x}} = \frac{1}{x}$$

so $u_2 = \int \frac{1}{x} dx = \ln|x| + c_2$

& $y = (-x + c_1)e^x + (\ln|x| + c_2) x e^x$

$y = c_1 e^x + c_3 x e^x + x e^x \ln|x|$] (Note: $c_3 = c_2 - 1$)

and the Wronskian

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^x & xe^x \\ e^x & (x+1)e^x \end{vmatrix}$$

$$= (x+1)e^{2x} - e^{2x} = e^{2x}.$$

Therefore

$$u_1(x) = -6\int \frac{(x^4 e^x)(xe^x)}{e^{2x}}\,dx$$

$$= -6\int x^5\,dx = -x^6,$$

and

$$u_2(x) = 6\int \frac{(x^4 e^x)(e^x)}{e^{2x}}\,dx$$

$$= 6\int x^4\,dx = \frac{6}{5}x^5.$$

Thus $y_p(x) = u_1 y_1 + u_2 y_2$

$$= -x^6 e^x + \frac{6}{5}x^5(xe^x) = \frac{1}{5}x^6 e^x.$$

The general solution of equation (1) can now be written as

$$y(x) = (c_1 + c_2 x)e^x + \frac{1}{5}x^6 e^x,$$

where c_1 and c_2 are arbitrary constants.

2-94

Let $x^2y'' - xy' + y = x\ln x$ in the interval $x > 0$.

a. Verify that $y_1 = x$ and $y_2 = x\ln x$ are solutions of the corresponding homogeneous differential equation.

b. Show that y_1 and y_2 form a fundamental set of solutions of the homogeneous differential equation on $(0,\infty)$.

c. Use the variation of parameters formula to obtain a particular solution y_p of the nonhomogeneous D. E.

d. Find the general solution of the given D. E.

****

a. $y_1 = x;\ y_1' = 1;\ y_1'' = 0.\quad x^2\cdot 0 - x\cdot 1 + x = 0.$

$y_2 = x\ln x;\ y_2' = 1+\ln x;\ y_2'' = 1/x.$

$x^2(1/x) - x(1+\ln x) + x\ln x = 0.$

b. It suffices to show that the Wronskian $W(y_1, y_2)$ is nonzero on the interval $(0,\infty)$.

$$W(y_1,y_2) = \begin{vmatrix} x & x\ln x \\ 1 & 1+\ln x \end{vmatrix} = x + x\ln x - x\ln x$$

$= x \neq 0$ for $x \in (0,\infty)$.

c. For the variation of parameters formula the standard form of the D.E. is $y'' + p(x)y' + q(x)y = g(x)$.

For this problem, we have $g(x) = \dfrac{x\ln x}{x^2}$ or $g(x) = \ln x / x$ (and <u>not</u> $x\ln x$).

$$y_p(x) = -y_1(x)\int^x \frac{y_2(t)}{W(y_1,y_2)(t)}g(t)\,dt + y_2(x)\int^x \frac{y_1(t)}{W(y_1,y_1)(t)}g(t)\,dt.$$

$$y_p(x) = -x\int^x \frac{t\ln t}{t}\cdot\frac{\ln t}{t}\,dt + x\ln x\int^x \frac{t}{t}\cdot\frac{\ln t}{t}\,dt$$

$$= -x\int^x \frac{\ln^2 t}{t}\,dt + x\ln x\int^x \frac{\ln t}{t}\,dt$$

2-93

Find the general solution: $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = \frac{e^{2x}}{x^2 + 3x}$

****

THE CHARACTERISTIC EQUATION IS $m^2 - 4m + 4 = (m-2)^2 = 0$, SO WE HAVE $y_c = c_1 e^{2x} + c_2 x e^{2x} = (c_1 + c_2 x) e^{2x}$.

WE APPLY THE METHOD OF VARIATION OF PARAMETERS TO FIND y_p:

FOR AN EQUATION OF THE FORM $a_0 y'' + a_1 y' + a_2 y = F(x)$

WITH $y_c = c_1 y_1 + c_2 y_2$, WE HAVE $y_p = v_1 y_1 + v_2 y_2$

WHERE $v_1 = \int \frac{-y_2 \cdot F(x)}{a_0 \cdot \omega(y_1, y_2)} dx$ AND $v_2 = \int \frac{y_1 \cdot F(x)}{a_0 \cdot \omega(y_1, y_2)} dx$.

THUS FOR $y_1 = e^{2x}$ AND $y_2 = x e^{2x}$,

$$\omega(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^{2x} & x e^{2x} \\ 2e^{2x} & (2x+1)e^{2x} \end{vmatrix} = e^{4x}.$$

$$v_1 = \int \frac{-x e^{2x} \cdot \frac{e^{2x}}{x^2+3x}}{1 \cdot e^{4x}} dx = \int \frac{-x\,dx}{x(x+3)} = \int \frac{-dx}{x+3}$$

$$= -\ln|x+3|$$

$$v_2 = \int \frac{e^{2x} \cdot \frac{e^{2x}}{x^2+3x}}{1 \cdot e^{4x}} dx = \int \frac{dx}{x(x+3)} = \int \left(\frac{1/3}{x} - \frac{1/3}{x+3} \right) dx$$

$$= \tfrac{1}{3}\ln|x| - \tfrac{1}{3}\ln|x+3| = \tfrac{1}{3}\ln\left|\frac{x}{x+3}\right|$$

$$\therefore\ y_p = v_1 y_1 + v_2 y_2 = -e^{2x}\ln|x+3| + \tfrac{1}{3} x e^{2x} \ln\left|\frac{x}{x+3}\right|$$

GENERAL SOLUTION:

$$y = y_c + y_p = (c_1 + c_2 x) e^{2x} + \left(\tfrac{1}{3} x \ln\left|\frac{x}{x+3}\right| - \ln|x+3| \right) e^{2x}$$

where $W[y_1, y_2] = \det \begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix} \equiv y_1 y_2' - y_2 y_1'$

is the Wronskian of y_1 and y_2.

$y'' - \frac{2}{x} y' + \frac{2}{x^2} y = \frac{1}{x^3}$ is then our eq with

$q(x) = -\frac{2}{x}$ and $b(x) = \frac{2}{x^2}$ and $f(x) = \frac{1}{x^3}$.

$W[y_1, y_2] = \det \begin{bmatrix} x & x^2 \\ 1 & 2x \end{bmatrix} = 2x^2 - x^2 = x^2.$

Using this info we can compute c_1 and c_2.

$$c_1' = -\frac{\frac{1}{x^3} \cdot x^2}{x^2} = -\frac{1}{x^3}$$

$$c_1 = \int (-\tfrac{1}{x^3})\, dx = -\left(\tfrac{1}{-2}\right) x^{-2} = \frac{1}{2x^2}$$

$$c_2' = \frac{\frac{1}{x^3} \cdot x}{x^2} = \frac{1}{x^4}$$

$$c_2 = \int x^{-4}\, dx = -\tfrac{1}{3} x^{-3} = -\frac{1}{3x^3}$$

$$y_p = c_1(x) y_1 + c_2(x) y_2 = \left(\frac{1}{2x^2}\right)(x) + \left(-\frac{1}{3x^3}\right)(x^2)$$

$$y_p = \frac{1}{2x} - \frac{1}{3x}$$

$$y_p = \frac{1}{6x}.$$

$$= -x\int^x \ln^2 t\, d(\ln t) + x\ln x \int^x \ln t\, d(\ln t)$$

$$= -x\,\frac{\ln^3 x}{3} + x\ln x\,\frac{\ln^2 x}{2}$$

$$= \frac{1}{6}x\ln^3 x = \frac{1}{6}x(\ln x)^3$$

d. $y = C_1 y_1 + C_2 y_2 + y_p$

$= C_1 x + C_2 x\ln x + \frac{1}{6}x\ln^3 x.$

2-95

Find a particular solution to $x^2y'' - 2xy' + 2y = \frac{1}{x}$, $x > 0$, given that $y_1 = x$ and $y_2 = x^2$ are two linearly independent solutions to the corresponding homogeneous equation $x^2y'' - 2xy' + 2y = 0$.

****

We can not use the method of undetermined coefficients since this eq. is not constant coefficients and the right hand side is not of the form a polynomial, exponential or sine and cosines.

If the eq is rewritten as $y'' + a(x)y' + b(x)y = f(x)$, then $y_p = C_1(x)\,y_1 + C_2(x)\,y_2$ where $C_1(x)$ and $C_2(x)$ satisfy

$$C_1' = -\frac{f(x)\,y_2}{W[y_1, y_2]} \quad \text{and} \quad C_2' = \frac{f(x)\,y_1}{W[y_1, y_2]}$$

2-97

Solve the differential equation $y' + xy = x$ by the Method of Variation of Parameters.

****

The corresponding homogeneous equation is solved as follows.

$$\frac{dy}{dx} + xy = 0$$

$$\frac{dy}{y} = -x\,dx \quad \text{as variables are separable}$$

$$\ln y = -\frac{x^2}{2} + c \quad \text{and} \quad y = e^{-x^2/2 + c}$$

Choose $c_1 = e^c$ and obtain $y = c_1 e^{-x^2/2}$

Assume a particular solution of the nonhomogeneous differential equation as $y = c(x)e^{-x^2/2}$.

For $y = c(x)e^{-x^2/2}$ we obtain $\frac{dy}{dx} = c'(x)e^{-x^2/2} - x\,c(x)e^{-x^2/2}$

Substituting these expressions into the differential equation $y' + xy = x$ we have

$$c'(x)e^{-x^2/2} - x\,c(x)e^{-x^2/2} + x\,c(x)e^{-x^2/2} = x$$

$$c'(x) = x e^{x^2/2}$$

Integration provides $c(x) = e^{x^2/2}$.

The solution of the differential equation may now be written

$$y = c(x)e^{-x^2/2}, c_1 e^{-x^2/2}$$

$$y = e^{x^2/2}, e^{-x^2/2}, c_1 e^{-x^2/2}$$

$$y = 1 + c_1 e^{-x^2/2}$$

2-96

Solve, using "Variation of Parameters": $y'' + 4y = \cos 2x$.

****

y_c SOLVES $y'' + 4y = 0$ SO $y_c = C_1 \cos 2x + C_2 \sin 2x$.

WE CHOOSE $y_p = A(x)\cos 2x + B(x)\sin 2x$.

$\therefore y_p' = -2A\sin 2x + 2B\cos 2x + A'\cos 2x + B'\sin 2x$.

THE 1ST CONSTRAINT ON A AND B IS THAT $A'\cos 2x + B'\sin 2x = 0$.

$\therefore y_p'' = -4A\cos 2x - 4B\sin 2x - 2A'\sin 2x + 2B'\cos 2x$.

$y_p'' + 4y_p = -2A'\sin 2x + 2B'\cos 2x$. THE 2ND CONSTRAINT ON A AND B IS THAT $y_p'' + 4y_p = \cos 2x$, i.e.,

$-2A'\sin 2x + 2B'\cos 2x = \cos 2x$. THIS GIVES US A SYSTEM:

$$\left.\begin{aligned} A'\cos 2x + B'\sin 2x &= 0 \\ -2A'\sin 2x + 2B'\cos 2x &= \cos 2x \end{aligned}\right\}$$ FROM THIS, $A' = -\frac{1}{2}\sin 2x\cos 2x$, $B' = \frac{1}{2}\cos^2 2x$.

$\therefore A = -\frac{1}{8}\sin^2 2x$ AND $B = \frac{x}{4} + \frac{1}{8}\sin 2x\cos 2x$.

$\therefore y_p = A\cos 2x + B\sin 2x = -\frac{1}{8}\sin^2 2x\cos 2x + \frac{x}{4}\sin 2x + \frac{1}{8}\sin^2 2x\cos 2x$

$= \frac{x}{4}\sin 2x$.

$y = y_c + y_p = C_1\cos 2x + C_2\sin 2x + \frac{x}{4}\sin 2x$.

A PASSING NOTE: USING C.A. HUTCHINSON'S OPERATOR TECHNIQUES, WE COULD WRITE: $y'' + 4y = \cos 2x$.

$y_c = C_1\cos 2x + C_2\sin 2x$. $y_p = \frac{1}{D^2+4}\cos 2x = \frac{x}{4}\sin 2x$.

FROM THE GENERAL FORM $\frac{1}{D^2+A^2}\cos Ax = \frac{x}{2A}\sin 2x$.

IT IS AN OLD BUT QUICK TECHNIQUE.

The solution of the corresponding homogeneous eq. is $y(x) = A\,e^{2x} + B\,e^{5x}$, where A and B are arbitrary constants.

So we assume that the general solution of the given nonhomogeneous eq. is

$$y(x) = A(x)\,e^{2x} + B(x)\,e^{5x}$$, where A(x) and B(x) are unknown functions of x and we have to find them.

We obtain the subsidiary equations

$$A'(x)\,e^{2x} + B'(x)\,e^{5x} = 0 \quad \ldots\ldots i$$

$$2A'(x)\,e^{2x} + 5B'(x)\,e^{5x} = e^{2x} \quad \ldots\ldots ii$$

Solving i and ii we get

$$A'(x) = -\tfrac{1}{3}, \quad B'(x) = \frac{e^{-3x}}{3}$$

Integrating $A(x) = C' - \tfrac{1}{3}x$, $B(x) = C_2 - \frac{e^{-3x}}{9}$

Therefore the general solution of the given nonhomogeneous equation is

$$y(x) = (C' - \tfrac{1}{3}x)\,e^{2x} + \left(C_2 - \frac{e^{-3x}}{9}\right)e^{5x}$$

$$= C'e^{2x} - \frac{1}{3}x\,e^{2x} + C_2 e^{5x} - \frac{1}{9}e^{2x}$$

$$y(x) = c_1 e^{2x} + c_2 e^{5x} - \tfrac{1}{3}x\,e^{2x}, \qquad c_1 = C' - \frac{1}{9}$$

where c_1, c_2 are arbitrary constants.

MISCELLANEOUS TECHNIQUES

2-100

Determine a homogenous linear differential equation with constant coefficients which has a general solution given by

$$y = c_o + c_1 e^{-x} + c_2 x e^{-x} + c_3 e^{-2x} + e^{-x}(c_4 \cos 2x + c_5 \sin 2x)$$

****

2-98

Find the general solution of the equation

$$y'' - 6y' + 8y = 3xe^{4x}$$

****

The solution of the corresponding homogeneous eq. is $y(x) = A e^{2x} + B e^{4x}$, where A and B are arbitrary constants.

So we assume that the general solution of the given nonhomogeneous eq. is

$y(x) = A(x) e^{2x} + B(x) e^{4x}$, where $A(x)$ and $B(x)$ are unknown functions of x and we have to find them.

We obtain the subsidiary equations

$$A'(x) e^{2x} + B'(x) e^{4x} = 0 \quad \ldots\ldots i$$

$$2A'(x) e^{2x} + 4B'(x) e^{4x} = 3x e^{4x} \quad \ldots\ldots ii$$

Solving i and ii we obtain

$$A'(x) = -\frac{3}{2} x e^{2x}, \quad B'(x) = \frac{3}{2} x$$

Integrating $A(x) = C_1 - \frac{3}{4} x e^{2x} + \frac{3}{8} e^{2x}$, $B(x) = C' + \frac{3}{4} x^2$

Therefore the general solution of the given nonhomogeneous equation is

$$y(x) = \left(C_1 - \frac{3}{4} x e^{2x} + \frac{3}{8} e^{2x}\right) e^{2x} + \left(C' + \frac{3}{4} x^2\right) e^{4x}$$

$$= C_1 e^{2x} + C_2 e^{4x} + \frac{3}{4} x^2 e^{4x} - \frac{3}{4} x e^{4x} \quad \left(C_2 = C' + \frac{3}{8}\right)$$

$$y(x) = C_1 e^{2x} + C_2 e^{4x} + \frac{3}{4} e^{4x}(x^2 - x)$$

2-99

Find the general solution of the nonhomogeneous equation

$$y'' - 7y' + 10y = e^{2x}$$

****

The differential equation is of order two and thus has a solution space of dimension two. Since y_1 and y_2 are linearly independent, they span the space and constitute a basis.

2-102

Solve this equation. You may write solutions in terms of definite integrals for those integrations that are overly difficult.

$$y'''-y' = \sec^2 x$$

****

Solution: $(D^3-D)y = \sec^2 x \Rightarrow D(D^2-1)Y = \sec^2 x$

integrating: $(D^2-1)Y_p = \tan x \;/\; Y_c = c_1+c_2e^x+c_3e^{-x}$

By variation of parameters: $Y_p = Ae^x+Be^{-x}$

$\Rightarrow Y_p' = Ae^x-Be^{-x}+A'e^x+B'e^{-x}$, Let $A'e^x+B'e^{-x}=0$ (1).

Then $Y_p'' = Ae^x+Be^{-x}+A'e^x-B'e^{-x}$

Then $Y_p''-Y_p = \tan x \Rightarrow Ae^x+Be^{-x}+A'e^x+B'e^{-x}$
$\underline{-Ae^x-Be^{-x}}$

$A'e^x-B'e^{-x} = \tan x$ (2).

from (1) $A'e^x+B'e^{-x}=0$

$\Rightarrow A' = \frac{1}{2}e^{-x}\tan x \Rightarrow 2A'e^x = \tan x$

$\Rightarrow A = \frac{1}{2}\int_0^x e^{-\beta}\tan\beta\, d\beta$

Similarly $B = -\frac{1}{2}\int_0^x e^{\beta}\tan\beta\, d\beta$

$\Rightarrow Y = Y_c+Y_p = c_1+c_2e^x+c_3e^{-x}+\frac{e^x}{2}\int_0^x e^{-\beta}\tan\beta\, d\beta-\frac{e^{-x}}{2}\int_0^x e^{\beta}\tan\beta\, d\beta$

The form of the solution indicates that the roots of the auxiliary equation are $0, -1, -1, -2, -1 \pm 2i$. The auxiliary equation may be written as $m(m^2+2m+5)(m+2)(m+1)^2 = 0$. The factor m generates the root $m=0$, the factor m^2+2m+5 generates the conjugate imaginary roots $m = -1 \pm 2i$, the factor $(m+1)^2$ generates the double root -1, and $m+2$ yields $m = -2$.

The differential equation may be written as $D(D+2)(D+1)^2(D^2+2D+5)y = 0$. $(D \sim \frac{d}{dx})$

2-101

The differential equation given by $y'' - y' - 6 = 0$ has two solutions given by $y_1 = \exp(3x)$, $y_2 = \exp(-2x)$. Show that these two solutions are linearly independent and therefore constitute a basis for the solution space of the differential equation.

****

The Wronskian of the two functions is given by the determinant

$$W(y_1, y_2) = \begin{vmatrix} e^{3x} & e^{-2x} \\ 3e^{3x} & -2e^{-2x} \end{vmatrix}$$

$$W(y_1, y_2) = -2e^{x} - 3e^{x}$$

$$= -5e^{x}$$

Since $e^x \neq 0$ for any finite x, the Wronskian of the two functions does not equal zero and therefore are linearly independent.

$y'' = e^x u'' + 2e^x u' + e^x u$. Consequently,

$$L[e^x u] = a(e^x u'' + 2e^x u' + e^x u) + b(e^x u' + e^x u) + c e^x u$$

$$= e^x(au'' + (2a+b)u' + (a+b+c)u),$$

and $\tilde{L}[u] = a u'' + (2a+b)u' + (a+b+c)u$.

2-105

Solve the equation: $y''+4y'+5y = 3e^{-2x}\cos(x)$

****

from $m^2+4m+5=0 \Rightarrow m = -2 \pm i$

then $Y_c = c_1 e^{-2x}\cos x + c_2 e^{-2x}\sin x$, $Y = Y_c + Y_p$

$(D^2+4D+5)Y_p = 3e^{-2x}\cos x$ using the exponential shift:

$e^{2x}(D^2+4D+5)Y_p = 3\cos x$

$\Rightarrow [(D-2)^2+4(D-2)+5](Y_p e^{2x}) = 3\cos x$

$\Rightarrow [D^2-4D+4+4D-8+5](Y_p e^{2x}) = 3\cos x$

$\Rightarrow (D^2+1)(Y_p e^{2x}) = 3\cos x$

$\Rightarrow Y_p e^{2x} = 3\dfrac{\cos x}{D^2+1} = 3\dfrac{x}{2}\sin x$

$\Rightarrow Y_p = \dfrac{3}{2}e^{-2x} x \sin x$

$\Rightarrow Y = c_1 e^{-2x}\cos x + c_2 e^{-2x}\sin x + \dfrac{3}{2}x e^{-2x}\sin x$

2-103

Suppose $L_3y = y''' + a_1y'' + a_2y' + a_3y = 0$, $a_i \in R$.

If 1, t, t^2 are solutions of $L_3y = 0$, solve the differential equation

$$L_3y = y''' + a_1y'' + a_2y' + a_3y = t.$$

****

Since $1, t, t^2$ are solutions:

$L_3(1) = 0 + 0 + 0 + a_3(1) = 0 \Rightarrow a_3 = 0$

$L_3(t) = 0 + 0 + a_2 \cdot 1 + 0 = 0 \Rightarrow a_2 = 0$

$L_3(t^2) = 0 + a_1(2) + 0 + 0 = 0 \Rightarrow a_1 = 0$

$\therefore L_3y = y''' = t$

We have 3 linearly independent solutions of the homogeneous d.e. We need a particular solution.

Try $y_p = t^3(A + Bt) = At^3 + Bt^4$

$y_p' = 3At^2 + 4Bt^3$

$y_p'' = 6At + 12Bt^2$

$y_p''' = 6A + 24Bt$

then $Ly_p = (6A + 24Bt) + 0 + 0 + 0 = t$

t^0: $6A = 0 \Rightarrow A = 0$

t^1: $24B = 1 \Rightarrow B = \frac{1}{24}$

$\therefore y_p = \frac{1}{24}t^4$

Hence the general solution is given by

$$y = C_1 \cdot 1 + C_2 \cdot t + C_3 \cdot t^2 + \frac{1}{24}t^4$$

2-104

Suppose $L[y] = ay'' + by' + cy$. Set $y = e^xu$, and show that

$L[e^xu] = e^x \tilde{L}[u]$, where

$\tilde{L}[u] = au'' + \_\_\_\_ u' + \_\_\_\_ u$. (fill in constants)

****

If $y = e^xu$, $y' = e^xu' + e^xu$, and

2-108

Use the Euler Tangent Line Method to estimate the value of the solution at $x = 0.3$ (i.e., the approximate value of $y(0.3)$) where y is the solution of the initial value problem $y' = y^2 - x^2$; $y(0) = 1$. Use step size $h = 0.1$.

****

$y' = y^2 - x^2 = f(x, y)$.

n	x_n	y_n	$y_n' = f(x_n, y_n)$	$y_{n+1} = y_n + h y_n'$
0	0	1	1	1.1
1	.1	1.1	1.2	1.22
2	.2	1.22	1.4484	1.36484
3	.3	1.36484		

$y(.3) \approx 1.36484$.

2-109

Suppose y_1 and y_2 are solutions to the first order equation $y' + p(x)y = g(x)$. Show that the Wronskian is given by

$$W(y_1, y_2)(x) = g(x)(y_1(x) - y_2(x)).$$

****

$$\begin{aligned} W(y_1, y_2)(x) &= y_1(x)\, y_2'(x) - y_1'(x)\, y_2(x) \\ &= y_1(x)\left[g(x) - p(x)y_2(x)\right] - y_2(x)\left[g(x) - p(x)y_1(x)\right] \\ &= y_1(x)g(x) - y_1(x)p(x)y_2(x) - y_2(x)g(x) + y_2(x)p(x)y_1(x) \\ &= g(x)\left(y_1(x) - y_2(x)\right). \end{aligned}$$

2-106

Given: $L[y] = y'' + by' + cy$; $L[y_1] = L[y_2] = 0$; $W(y_1, y_2) = e^{3x}$. What is b?

If, additionally, $y_1 = e^x$: What is c? What can be said about y_2?

****

Since $W' + bW = 0$, and $W' = 3e^{3x}$, it must be that $\underline{b = -3}$.

To find c, note that $L[y_1] = e^x - 3e^x + ce^x$, so $\underline{c = 2}$. $L[y] = y'' - 3y' + 2y = 0$ has as its characteristic equation $r^2 - 3r + 2 = 0$, and $r^2 - 3r + 2 = (r-1)(r-2)$. As y_2 is a solution, $y_2 = ke^x + \ell e^{2x}$, and

$$W(y_1, y_2) = e^x(ke^x + 2\ell e^{2x}) - e^x(ke^x + \ell e^{2x})$$
$$= \ell e^{3x}, \text{ so } \ell = 1 \text{ and } \underline{y_2 = ke^x + e^{2x}}.$$

2-107

Are the functions 7, sinx, $\sin^2 x$, and cos2x linearly independent? Support your answer.

****

Recall the trigonometric identity $\cos 2x = 1 - 2\sin^2 x$

so $1 - 2\sin^2 x - \cos 2x = 0$

or $\frac{1}{7}(7) - 2\sin^2 x - \cos 2x = 0$

The functions 7, sinx, $\sin^2 x$, and cos2x are linearly dependent, since a nontrivial linear combination adds to zero.

2-111

Let D be the operator $\frac{d}{dx}$ and f a polynomial of degree n.

(a) If a is a complex number and $f(a) \neq 0$, then show that

$$[f(D)]^{-1}.e^{ax} = (1/f(a))e^{ax}.$$

(b) Use (a) to find particular solutions of (i) $y'' - 2y' + 3y = e^{2x}$, and

(ii) $y'' - 2y' + 3y = 17\sin 2x$.

****

(a) Let $f(x) = \sum_{k=0}^{n} c_k x^k$. Then $f(D) = \sum_{k=0}^{n} c_k D^k$.

$$D^k.e^{ax} = a^k e^{ax}, \quad k = 0, 1, \ldots, n$$

$$f(D).e^{ax} = \left[\sum_{k=0}^{n} c_k D^k\right].e^{ax}$$

$$= \sum_{k=0}^{n} c_k D^k.e^{ax}$$

$$= \sum_{k=0}^{n} c_k a^k e^{ax}$$

$$= f(a)e^{ax}.$$

If $f(a) \neq 0$, then

$$f(D).\left[\frac{1}{f(a)}e^{ax}\right] = \frac{1}{f(a)}f(D).e^{ax} = \frac{1}{f(a)}.f(a)e^{ax} = e^{ax}.$$

Thus $[f(D)]^{-1}.e^{ax} = \frac{1}{f(a)}e^{ax}$.

(b) (i) The equation can be written as

$$(D^2-2D+3).y = e^{2x}$$

$\therefore$ $y = (D^2-2D+3)^{-1}.e^{2x}$ is a particular solution,

i.e. $y = \frac{1}{2^2-2(2)+3}e^{2x}$

$y = \frac{1}{3}e^{2x}$ is a particular solution.

2-110

Let D be the operator $\frac{d}{dx}$, f a polynomial of degree n and g an n-times differentiable function.

(a) Show that $[f(D)]^{-1}.\ e^{ax}g(x) = e^{ax}[f(D+a)]^{-1}.g(x)$.

(b) Use (a) to find a particular solution of $y'' - 2y' + y = e^x$.

****

(a) First we will show that $D^k.[e^{ax}g(x)] = e^{ax}(D+a)^k.\ g(x)$.

Proof: Suppose $D^k.[e^{ax}g(x)] = e^{ax}(D+a)^k.\ g(x)$, $k<n$.

Then
$$\begin{aligned} D^{k+1}.[e^{ax}g(x)] &= D^k.[D.(e^{ax}g(x))] \\ &= D^k.[e^{ax}D.g(x)+ae^{ax}g(x)] \\ &= D^k.[e^{ax}(D+a).g(x)] \\ &= e^{ax}(D+a)^k.[(D+a).g(x)] \\ &= e^{ax}(D+a)^{k+1}.g(x). \end{aligned}$$

$$\begin{aligned} D^0.[e^{ax}g(x)] &= e^{ax}g(x) \\ &= e^{ax}(D+a)^0.g(x) \end{aligned}$$

By induction $D^k.[e^{ax}g(x)] = e^{ax}(D+a)^k.\ g(x)$, for $k=0,1,2,\ldots,n$

If $f(D) = \sum_{k=0}^{n} c_k D^k$, then it follows that

$$f(D).[e^{ax}g(x)] = e^{ax}f(D+a).g(x).$$

Let $h(x) = [f(D+a)]^{-1}.\ g(x)$. Then

$$\begin{aligned} f(D).[e^{ax}h(x)] &= e^{ax}f(D+a).h(x) \\ &= e^{ax}f(D+a).\{[f(D+a)]^{-1}.g(x)\} \\ &= e^{ax}g(x) \end{aligned}$$

$$\therefore\ [f(D)]^{-1}.\ e^{ax}g(x) = e^{ax}[f(D+a)]^{-1}.g(x).$$

(b) The equation is $(D^2-2D+1)y = e^x$.

$$\begin{aligned} \therefore\ y &= (D^2-2D+1)^{-1}.[e^x.1] \\ &= e^x[(D+1)^2-2(D+1)+1]^{-1}.1 \\ &= e^x[D^2]^{-1}.1 = e^xD^{-1}.(D^{-1}.1) \\ &= e^xD^{-1}.x = e^x.\frac{x^2}{2} = \frac{1}{2}x^2e^x. \end{aligned}$$

2-113

$y'' + 2y' + 5y = e^{-x}\cos 2x$. A particular solution has the form

(A) $C_1 x e^{-x}\cos 2x$

(B) $C_1 x e^{-x}\cos 2x + C_2 x e^{-x}\sin 2x$

(C) $C_1 e^{-x}\cos 2x + C_2 e^{-x}\sin 2x$

(D) $C_1 e^{-x}\cos 2x$

(E) $C_1 e^{-x}\cos 2x + C_2 x e^{-x}\cos 2x$

****

Denoting $D = \frac{d}{dx}$, $(D^2 + 2D + 5)y = e^{-x}\cos 2x$

The operator may be decomposed as follows:

$$D^2 + 2D + 5 = (D+1)^2 + 2^2 = (D+1-2i)(D+1+2i).$$

The right hand side may be written as

$$e^{-x}\cos 2x = e^{-x}\,\tfrac{1}{2}\left[e^{2ix} + e^{-2ix}\right] = \tfrac{1}{2}\underbrace{e^{-x}e^{2ix}}_{e^{(-1+2i)x}} + \tfrac{1}{2}\underbrace{e^{-x}e^{-2ix}}_{e^{(-1-2i)x}}$$

The equation is

$$(*)\quad (D+1-2i)(D+1+2i)\,y = \tfrac{1}{2}e^{(-1+2i)x} + \tfrac{1}{2}e^{(-1-2i)x}.$$

We seek an equation that equals zero. To that end we note that $e^{(-1+2i)x}$ is annihilated by the operator $D+1-2i$ i.e. $(D+1-2i)\,e^{(-1+2i)x} = 0$. $e^{(-1-2i)x}$ is annihilated by $D+1+2i$. To annihilate the sum $e^{(-1+2i)x} + e^{(-1-2i)x}$ we need the composition operator. Indeed, applying $(D+1-2i)(D+1+2i)$ to both sides of the equation (*), the right hand side of the equation vanishes since

b (ii) The equation can be written as

$$(D^2-2D+3).\ y = 17\sin 2x$$

$$\therefore \quad y = (D^2-2D+3)^{-1}.\ 17\sin 2x$$

$$= (D^2-2D+3)^{-1}.\ 17\ \mathrm{Im}(e^{2ix})$$

$$= 17\ \mathrm{Im}\left[(D^2-2D+3)^{-1}.\ e^{2ix}\right]$$

$$= 17\ \mathrm{Im}\left[\frac{1}{(2i)^2-2(2i)+3}\ e^{2ix}\right]$$

$$= 17\ \mathrm{Im}\left[\frac{\cos 2x+i\sin 2x}{-1-4i}\right]$$

$$= 17\ \mathrm{Im}\left[\frac{(\cos 2x+i\sin 2x)(-1+4i)}{1^2+4^2}\right]$$

$$= 17\ \mathrm{Im}\left[\frac{(-\cos 2x-4\sin 2x)+i(4\cos 2x-\sin 2x)}{17}\right]$$

$$= 4\cos 2x - \sin 2x.$$

2-112

Solve the equation: $(D-2)(D+3) = 2 + 3e^x$

****

Solution: The general solution will be of the form $Y = Y_c + Y_p$ where $Y_c = c_1e^{2x} + c_2e^{-3x}$

by inspection $Y_p = \frac{2}{-6} + \frac{3e^x}{F(1)}$ for $F(D) = (D-2)(D+3)$

$$\Rightarrow Y_p = \frac{-1}{3} + \frac{3e^x}{(1-2)(1+3)} = \frac{-1}{3} + \frac{3e^x}{-1(4)}$$

$$\Rightarrow Y = c_1e^{2x} + c_2e^{-3x} - \frac{1}{3} - \frac{3}{4}e^x$$

2-114

(a) Show that two solutions of the equation

$$x^2y'' + xy' - \alpha^2 y = 0, \ \alpha \text{ constant}, \ x > 0$$

are $y_1 = x^{\alpha}$ and $y_2 = x^{-\alpha}$.

(b) Compute their Wronskian, and hence show that the solutions are linearly independent, except for one value of α. Write the general solution when y_1 and y_2 are linearly independent.

(c) What is the general solution for the exceptional value of α?

****

(a) This is an Euler DE, set $y=x^r, y'=rx^{r-1}, y''=r(r-1)x^{r-2}$

$$x^2y''+xy'-\alpha^2y = x^2[r(r-1)x^{r-2}] + x[rx^{r-1}] - \alpha^2x^r = 0$$

Since $x \neq 0 \Rightarrow r(r-1)+r-\alpha^2 = r^2-\alpha^2 = 0$

$\Rightarrow r = \pm\alpha$. Thus $y_1 = x^{\alpha}$ and $y_2 = x^{-\alpha}$

(b) $$W(y_1,y_2) = \begin{vmatrix} x^{\alpha} & x^{-\alpha} \\ \alpha x^{\alpha-1} & -\alpha x^{-\alpha-1} \end{vmatrix} = -\alpha x^{-1} - \alpha x^{-1} = -2\alpha x^{-1}$$

So $W \neq 0$, unless $\alpha = 0$.

Gen. Solution $\boxed{y = c_1x^{\alpha} + c_2x^{-\alpha}, \ \alpha \neq 0}$

(c) When $\alpha = 0$; $r = 0, 0$ is a repeated root,

Then $\boxed{y = c_1 + c_2 \ln x, \ \alpha = 0}$

$$\underbrace{\frac{(D+1-2i)(D+1+2i)}{(D+1+2i)(D+1-2i)}}_{=0}e^{(-1+2i)x} + \underbrace{(D+1-2i)(D+1+2i)e^{(-1-2i)x}}_{=0} = 0$$

and the equation becomes

$$(D+1-2i)^2(D+1+2i)^2 y = 0$$

To solve it put $y = e^{mx}$. Then, $(m+1-2i)^2(m+1+2i)^2 e^{mx} = 0$.

Since e^{mx} is not zero, we must have

$$(m+1-2i)^2(m+1+2i)^2 = 0$$

The solutions are: $m_1 = m_2 = -1+2i$; $m_3 = m_4 = -1-2i$;

Hence, the general solution to the differential equation has the form

$$C_1 e^{(-1+2i)x} + C_2 x e^{(-1+2i)x} + C_3 e^{(-1-2i)x} + C_4 x e^{(-1-2i)x},$$

where the C's are constants. The particular solution is the part that is not annihilated by the operator in the equation (*).

Here, the particular solution is

$$C_2 x \underbrace{e^{(-1+2i)x}}_{e^{-x}e^{2ix}} + C_4 x \underbrace{e^{(-1-2i)x}}_{e^{-x}e^{-2ix}} = xe^{-x}\left[C_2 \underbrace{e^{2ix}}_{\cos 2x + i\sin 2x} + C_4 \underbrace{e^{-2ix}}_{\cos 2x - i\sin 2x}\right]$$

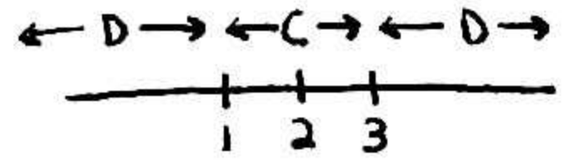
$$= xe^{-x}\left(\underbrace{[C_2 + C_4]}_{\text{some constant call it } a_1}\cos 2x + i\underbrace{[C_2 - C_4]}_{\text{some constant : } a_2}\sin 2x\right)$$

$$= a_1 x e^{-x}\cos 2x + a_2 x e^{-x}\sin 2x .$$

The correct answer is (B).

Use the Ratio Test to determine the radius of convergence.

$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(-1)^{n+1}(x-2)^{n+1}}{(-1)^n(x-2)^n}\right| = |x-2|$, so the series converges if $|x-2|<1$, diverges if $|x-2|>1$.



The graph shows regions of convergence (C) and divergence (D).

The radius of convergence is 1.

3-2

Determine the interval of convergence for the power series

$$\sum_{n=1}^{\infty} \frac{n(x-1)^n}{(2n+1)^3}$$

****

Using the ratio test $\lim_{n\to\infty} \frac{(n+1)(2n+1)^3}{n(2n+3)^3}|x-1| =$

$= |x-1|$. We have absolute convergence for $|x-1|<1$ or $0<x<2$.

For $x=0$, $\sum_{n=1}^{\infty} \frac{n(-1)^n}{(2n+1)^3}$ converges by the alternating series test

For $x=2$, $\sum_{n=1}^{\infty} \frac{n}{(2n+1)^3}$ converges by the limit comparison test using the p-series $\sum_{n=1}^{\infty} \frac{1}{n^2}$.

3
SOLUTION BY POWER SERIES

REVIEW OF POWER SERIES

3-1

Find the Taylor series expansion of $f(x) = 1/(x-1)$ about $x=2$, and determine the radius of convergence.

****

First, compute $f(2) = 1$; $f'(x) = -1/(x-1)^2$, so $f'(2) = -1$; $f''(x) = 2/(x-1)^3$, so $f''(2) = 2$; $f'''(x) = -3\cdot 2/(x-1)^4$, so $f'''(2) = -3!$; $f''''(x) = 4\cdot 3\cdot 2/(x-1)^5$, so $f''''(2) = 4!$; it is clear from the pattern that $f^{(n)}(2) = (-1)^n\, n!$, where the $(-1)^n$ takes care of the alternating +/- signs.

The Taylor series form is $f(x) = \sum_{n=0}^{\infty} f^{(n)}(2)\, \frac{(x-2)^n}{n!}$, so

$$\frac{1}{x-1} = \sum_{n=0}^{\infty} (-1)^n\, n!\, \frac{(x-2)^n}{n!} = \sum_{n=0}^{\infty} (-1)^n (x-2)^n .$$

3-4

Show that $y = \sum_{n=0}^{\infty} \frac{(-1)^n}{4^n (n!)^2} \cdot x^{2n}$ is a solution to $xy'' + y' + xy = 0$.

****

We can differentiate y term-by-term, rearrange the resulting terms and collect coefficients of corresponding powers of x. All of the coefficients in the result should be zero.

$$y = \sum_{n=0}^{\infty} \frac{(-1)^n}{4^n (n!)^2} x^{2n}, \quad y' = \sum_{n=1}^{\infty} \frac{(-1)^n (2n)}{4^n (n!)^2} x^{2n-1}, \quad y'' = \sum_{n=1}^{\infty} \frac{(-1)^n (2n)(2n-1)}{4^n (n!)^2} x^{2n-2}$$

Substituting these into the left hand side of the DE gives $xy'' + y' + xy$

$$= x \sum_{n=1}^{\infty} \frac{(-1)^n (2n)(2n-1)}{4^n (n!)^2} x^{2n-2} + \sum_{n=1}^{\infty} \frac{(-1)^n (2n)}{4^n (n!)^2} x^{2n-1}$$

$$+ x \sum_{n=0}^{\infty} \frac{(-1)^n}{4^n (n!)^2} x^{2n}$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^n (2n)(2n-1)}{4^n (n!)^2} x^{2n-1} + \sum_{n=1}^{\infty} \frac{(-1)^n (2n)}{4^n (n!)^2} x^{2n-1}$$

$$+ \sum_{n=0}^{\infty} \frac{(-1)^n}{4^n (n!)^2} x^{2n+1}$$

3-3

For which values of x does the series $\sum_{k=1}^{\infty} \frac{x^k}{k^2}$ converge?

****

For this series $a_k = \frac{x^k}{k^2}$ and $a_{k+1} = \frac{x^{k+1}}{(k+1)^2}$

By the ratio test we have

$$r = \lim_{k\to\infty} \left| \frac{a_{k+1}}{a_k} \right|$$

$$= \lim_{k\to\infty} \left| \frac{\frac{x^{k+1}}{(k+1)^2}}{\frac{x^k}{k^2}} \right| = \lim_{k\to\infty} \left| \frac{k^2}{(k+1)^2} x \right|$$

$$= \lim_{k\to\infty} \left| \frac{k^2}{(k+1)^2} \right| |x|$$

Note: $\frac{k^2}{(k+1)^2} = \frac{k^2}{k^2+2k+1} = \frac{1}{1+\frac{2}{k}+\frac{1}{k^2}}$

$$= \lim_{k\to\infty} |x|$$

Therefore series converges for $|x|<1$. For $|x|=1$ the ratio test fails. But for $x=1$ the series becomes

$$\sum_{k=1}^{\infty} \frac{1}{k^2} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots$$

which is the p series with $p=2$ and known to converge.

Thus series converges for $|x| \leq 1$.

3-5

Find power series expansion of ln(1+x) near x=0.

****

Differentiating $\ln(1+x)$, we obtain $\frac{1}{1+x}$. We are familiar with the series expansion

$$\frac{1}{1-x} = 1 + x + x^2 + \cdots + x^n + \cdots \qquad (1)$$

Replace x by $-x$ in (1):

$$\frac{1}{1+x} = 1 - x + x^2 + \cdots + (-1)^n x^n + \cdots \qquad (2)$$

Integrating both side of (2), we find

$$\ln(1+x) = x - \tfrac{1}{2}x^2 + \tfrac{1}{3}x^3 + \cdots + \tfrac{1}{n+1}(-1)^n x^{n+1} + \cdots$$

This is the desired expansion.

In the 1st term let $n = k+1$, in the 2nd let $n = k+1$ and in the 3rd let $n = k$.

$$\sum_{k=0}^{\infty} \frac{(-1)^{k+1}\, 2(k+1)(2k+1)}{4^{k+1}((k+1)!)^2} x^{2k+1} + \sum_{k=0}^{\infty} \frac{(-1)^{k+1}\, 2(k+1)}{4^{k+1}((k+1)!)^2} x^{2k+1}$$

$$+ \sum_{k=0}^{\infty} \frac{(-1)^k}{4^k (k!)^2} x^{2k+1}$$

$$= \sum_{k=0}^{\infty} \left[\frac{(-1)^{k+1}\, 2(k+1)(2k+1)}{4^{k+1}((k+1)!)^2} + \frac{(-1)^{k+1}\, 2(k+1)}{4^{k+1}((k+1)!)^2} + \frac{(-1)^k}{4^k (k!)^2} \right] x^{2k+1}$$

$$= \sum_{k=0}^{\infty} \left[\frac{(-1)^{k+1}\, 2(k+1)(2k+1) + (-1)^{k+1}\, 2(k+1) + (-1)^k\, 4(k+1)^2}{4^{k+1}((k+1)!)^2} \right] x^{2k+1}$$

$$= \sum_{k=0}^{\infty} (-1)^{k+1}(k+1) \frac{(4k+2) + (2) - (4k+4)}{4^{k+1}((k+1)!)^2} x^{2k+1}$$

$$= \sum_{k=0}^{\infty} (-1)^{k+1}(k+1) \left[\frac{0}{4^{k+1}((k+1)!)^2} \right] x^{2k+1}$$

$$= \sum_{k=0}^{\infty} 0 = 0$$

Hence y is a solution to $xy'' + y' + xy = 0$.

3-7

Find a solution of the equation

$$y' - 3x^2 y = 0. \qquad (1)$$

in the form of a power series about $x = 0$.

* *

Let $$y(x) = \sum_{n=0}^{\infty} a_n x^n.$$

Then $$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

Substituting these into the differential equation, we have

$$\sum_{n=1}^{\infty} n a_n x^{n-1} - 3 \sum_{n=0}^{\infty} a_n x^{n+2} = 0. \qquad (2)$$

Shifting indices, we write

$$\sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n,$$

$$\sum_{n=0}^{\infty} a_n x^{n+2} = \sum_{n=2}^{\infty} a_{n-2} x^n$$

Equation (2) now becomes

$$\sum_{n=0}^{\infty} (n+1) a_{n+1} x^n - 3 \sum_{n=2}^{\infty} a_{n-2} x^n = 0$$

which may be written as

FIRST ORDER EQUATIONS

3-6

Use power series to find a solution of the equation (near $x=0$)

$$(1+2x)y' + 2y = 0 \qquad (1)$$

****

Let $y = \sum_{n=0}^{\infty} a_n x^n$. Then $y' = \sum_{n=0}^{\infty} a_n n x^{n-1}$,

$2xy' = \sum_{n=0}^{\infty} 2n a_n x^n$. y' can be written as

$\sum_{n=0}^{\infty} a_{n+1}(n+1) x^n$. Substitute all the above into (1)

$$\sum_{n=0}^{\infty} \left[a_{n+1}(n+1) + 2n a_n + 2a_n \right] x^n = 0$$

This implies all the coefficients of x^n $(n=0, 1, 2, \cdots)$ must equal to 0, $a_{n+1}(n+1) + 2n a_n + 2a_n = 0$,

i.e. $$a_{n+1} = -2a_n$$

This recursion formula enables us to find all the coefficients in terms of a_0: $a_n = -2a_{n-1} = \cdots = (-2)^n a_0$

Hence the series solution of (1) is $y = a_0 \sum_{n=0}^{\infty} (-2x)^n$.

We recognize that $\sum_{n=0}^{\infty} (-2x)^n$ is the power series expansion of function $\frac{1}{1+2x}$ near $x=0$.

3-8

Use power series methods to solve $y' + xy = x^2$.

****

Since the coefficients have power series expansions (being polynomials) so does y.
(This is true when the coefficient of y' is 1.)

Let $y = \sum_{n=0}^{\infty} a_n x^n$, then $y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$.

Substitute these into the DE.

$$\sum_{n=1}^{\infty} n a_n x^{n-1} + x \sum_{n=0}^{\infty} a_n x^n = x^2$$

Let $n = k+2$ ($k = n-2$) in the 1st term and $k = n$ in the 2nd.

$$\sum_{k=-1}^{\infty} (k+2) a_{k+2} x^{k+1} + \sum_{k=0}^{\infty} a_k x^{k+1} = x^2$$

$$(1)\ a_1 + \sum_{k=0}^{\infty} (k+2) a_{k+2} x^{k+1} + \sum_{k=0}^{\infty} a_k x^{k+1} = x^2$$

$$a_1 + \sum_{k=0}^{\infty} \left[(k+2) a_{k+2} + a_k \right] x^{k+1} = x^2$$

$$a_1 + (2a_2 + a_0) x + (3a_3 + a_1) x^2 + \sum_{k=2}^{\infty} \left[(k+2) a_{k+2} + a_k \right] x^{k+1} = x^2$$

Equate coefficients of corresponding powers of x on the left and right.

$$a_1 + 2a_2 x + \sum_{n=2}^{\infty} [(n+1)a_{n+1} - 3a_{n-2}]x^n = 0.$$

Since the above equation must hold for all x in some interval about the point $x=0$, every coefficient of x in this equation must vanish. It follows that $a_1 = a_2 = 0$ and we have the recurrence relation

$$(n+1)a_{n+1} - 3a_{n-2} = 0, \quad n = 2, 3, 4, \cdots$$

This gives

$$a_{n+1} = \frac{3}{n+1} a_{n-2}, \quad n = 2, 3, 4, \cdots$$

Using $n = 2, 3, 4, 5, \cdots$ successively, we find

$$a_3 = a_0, \quad a_4 = \frac{3}{4} a_1 = 0,$$

$$a_5 = \frac{3}{5} a_2 = 0, \quad a_6 = \frac{3}{6} a_3 = \frac{1}{2} a_0,$$

$$a_7 = a_8 = 0, \quad a_9 = \frac{1}{3} a_6 = \frac{1}{3.2} a_0,$$

$$a_{10} = a_{11} = 0, \quad a_{12} = \frac{1}{4} a_9 = \frac{1}{4.3.2} a_0$$

and so on. (Note that the constant a_0 is undetermined.)

Therefore the power series solution is:

$$y(x) = a_0 \left(1 + x^3 + \frac{x^6}{2!} + \frac{x^9}{3!} + \frac{x^{12}}{4!} + \cdots\right).$$

Note that equation (1) has an exact solution $y(x) = a_0 \exp(x^3)$ which expands into the power series obtained above.

3-9

Solve the differential equation using power series:

$$(1 + x^2)y' = 6xy \qquad \text{where } y(0) = 1.$$

****

Assume $y = \sum_{n=0}^{\infty} c_n x^n$

$y' = \sum_{n=1}^{\infty} c_n \cdot n \cdot x^{n-1}$, substituting into the d.e. yields $(1+x^2) \sum_{n=1}^{\infty} n c_n x^{n-1} - 6x \sum_{n=0}^{\infty} c_n x^n = 0$

$\Rightarrow \sum_{n=1}^{\infty} n c_n x^{n-1} + \sum_{n=1}^{\infty} n c_n x^{n+1} - 6 \sum_{n=0}^{\infty} c_n x^{n+1} = 0$, rewriting:

let $K = n-1$, $K+1 = n$; let $K = n+1$, $K-1 = n$; let $K = n+1$, $K-1 = n$

$$\sum_{K=0}^{\infty} (K+1) c_{K+1} x^K + \sum_{K=2}^{\infty} (K-1) c_{K-1} x^K - 6 \sum_{K=1}^{\infty} c_{K-1} x^K = 0$$

x^0: $\quad 1 \cdot c_1 = 0 \Rightarrow c_1 = 0$

x^1: $\quad 2 \cdot c_2 - 6 c_0 = 0 \Rightarrow c_2 = 3 c_0$

x^2: $\quad 3 \cdot c_3 + 1 \cdot c_1 - 6 c_1 = 0 \Rightarrow c_3 = \frac{5}{3} c_1 = \frac{5}{3} \cdot 0 = 0$

In general, for $K \geq 2$:

x^K: $\quad (K+1) c_{K+1} + (K-1) c_{K-1} - 6 c_{K-1} = 0$

$\quad (K+1) c_{K+1} + (K-7) c_{K-1} = 0$

$$\therefore c_{K+1} = -\left(\frac{K-7}{K+1}\right) c_{K-1} \quad , K \geq 2$$

Since $c_1 = 0 \Rightarrow c_3 = c_5 = \cdots = 0$

and $c_4 = \frac{4}{4} c_2 = c_2 = 3c_0$

$c_6 = \frac{2}{6} c_4 = \frac{1}{3} c_4 = \frac{1}{3}(3c_0) = c_0$

$c_8 = 0 \cdot c_6 = 0$, $\therefore c_0 = c_{12} = \cdots = 0$ also,

$\therefore y = \sum_{n=0}^{\infty} c_n x^n = c_0 + 3c_0 x^2 + 3c_0 x^4 + c_0 x^6 = c_0(1 + 3x^2 + 3x^4 + x^6)$

and, in fact, this can be written as $y = c_0 (1+x^2)^3$

But $y(0) = 1 \Rightarrow \quad 1 = y(0) = c_0(1) \Rightarrow c_0 = 1 \therefore$ the solution is given by: $\underline{\underline{y = (1+x^2)^3}}$

a_0 is arbitrary

$a_1 = 0$ $\qquad a_1 = 0$

$2 \cdot a_2 + a_0 = 0$ $\qquad a_2 = -\frac{1}{2} a_0$

$3 a_3 + a_1 = 1$ $\qquad a_3 = \frac{1}{3}(1 - a_1) = \frac{1}{3}$

$(k+2) a_{k+2} + a_k = 0, \; k = 2, 3, 4, \cdots$ $\qquad a_{k+2} = -\frac{1}{k+2} a_k$

The last expression (the recurrence relation)

gives $k = 2$ $\qquad a_4 = -\frac{1}{4} a_2 = \frac{1}{2 \cdot 4} a_0$

$k = 3$ $\qquad a_5 = -\frac{1}{5} a_3 = -\frac{1}{3 \cdot 5}$

$k = 4$ $\qquad a_6 = -\frac{1}{6} a_4 = -\frac{1}{6} \cdot \frac{1}{2 \cdot 4} a_0 = -\frac{1}{2 \cdot 4 \cdot 6} a_0$

Putting these into $y = \sum_{n=0}^{\infty} a_n x^n$ gives

$$y = a_0 - \frac{1}{2} a_0 x^2 + \frac{1}{3} x^3 + \frac{1}{2 \cdot 4} a_0 x^4 - \frac{1}{3 \cdot 5} x^5 - \frac{1}{2 \cdot 4 \cdot 6} a_0 x^6 + \cdots$$

$$y = a_0 \left(1 - \frac{1}{2} x^2 + \frac{1}{2 \cdot 4} x^4 - \frac{1}{2 \cdot 4 \cdot 6} x^6 + \cdots\right) + \left(\frac{1}{3} x^3 - \frac{1}{3 \cdot 5} x^5 + \cdots\right)$$

$$y = a_0 \left(1 - \frac{1}{2} x^2 + \frac{1}{2^2 \cdot 1 \cdot 2} x^4 - \frac{1}{2^3 \cdot 1 \cdot 2 \cdot 3} x^6 + \cdots\right) + \left(\frac{1}{3} x^3 - \frac{1}{3 \cdot 5} x^5 + \cdots\right)$$

$$y = a_0 \left(1 - \frac{1}{2} x^2 + \frac{1}{2^2 \cdot 2!} (x^2)^2 - \frac{1}{2^3 \cdot 3!} (x^2)^3 + \cdots\right) + \left(\frac{1}{3} x^3 - \frac{1}{3 \cdot 5} x^5 + \cdots\right)$$

$$y = a_0 \left(1 - \left(\frac{x^2}{2}\right)^1 + \frac{1}{2!} \left(\frac{x^2}{2}\right)^2 - \frac{1}{3!} \left(\frac{x^2}{2}\right)^3 + \cdots\right) + \left(\frac{1}{3} x^3 - \frac{1}{3 \cdot 5} x^5 + \cdots\right)$$

$$y = a_0 e^{-\frac{x^2}{2}} + \left(\frac{1}{3} x^3 - \frac{1}{3 \cdot 5} x^5 + \cdots\right).$$

Therefore $y(x) = \sum_{n=0}^{\infty} a_{2n} x^{2n}$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{a_0}{2^n n!} x^{2n}$$

$$= a_0 \left(1 - \frac{1}{2}x^2 + \frac{1}{2!}\left(\frac{x^2}{2}\right)^2 - \frac{1}{3!}\left(\frac{x^2}{2}\right)^3 + \dots\right)$$

$$= a_0 e^{-x^2/2}$$

$y(0) = a_0 = 1$ gives $y(x) = e^{-x^2/2}$

To obtain the solution by integrating we observe that the equation is separable

$$\frac{dy}{dx} + xy = 0$$

$$\frac{dy}{y} + x\,dx = 0$$

Integrating we obtain $\ln|y| + \frac{x^2}{2} = C$. Apply the initial condition we get $\ln 1 + \frac{0^2}{2} = C$. That is $C = 0$.

Therefore $\ln|y| = -\frac{x^2}{2}$

$$y = e^{-x^2/2}, \quad y > 0$$

which is exactly the solution obtained by the power series method.

CLASSIFICATION OF THE BASE POINT

3-11

Consider the equation (near $x \simeq 0$)

$$x^2(x+1)y'' + xe^x y' - ((4/9)\cos x)y \simeq 0 \qquad (1)$$

Which of the following is most likely the general solution of the equation ?

3-10

Obtain a power series solution of the initial value problem

$$y' + xy = 0 , \quad y(0) = 1.$$

Compare with the solution obtained by integrating the equation.

****

Assume $y = \sum_{n=0}^{\infty} a_n x^n$, $y' = \sum_{n=0}^{\infty} n\, a_n x^{n-1}$

$$= \sum_{n=0}^{\infty} (n+1)\, a_{n+1} x^n$$

Substituting in the equation, we obtain

$$\sum_{n=0}^{\infty} (n+1)\, a_{n+1} x^n + x \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} (n+1)\, a_{n+1} x^n + \sum_{n=0}^{\infty} a_n x^{n+1} = 0$$

Since the RHS is zero, we set the coefficient of x^n equal to zero giving

$$(n+1)\, a_{n+1} + a_{n-1} = 0 \qquad n \geq 1$$

$$a_1 = 0 \qquad n = 0$$

Therefore $a_{n+1} = -\frac{1}{n+1} a_{n-1} \quad n \geq 1 \ldots . i$

From the recurrence formula i, we obtain

$$a_3 = a_5 = a_7 = \ldots\ldots\ldots = 0$$

Also $a_2 = -\frac{a_0}{2}$, $a_4 = -\frac{a_2}{4} = \frac{1}{2.4} a_0 = \frac{1}{2^2(2!)}$

$$a_6 = -\frac{a_4}{6} = -\frac{a_0}{2.4.6} = -\frac{a_0}{2^3(3!)}$$

$$a_{2n} = (-1)^n \frac{a_0}{2^n (n!)}$$

3-12

Find all the singular points for each differential equation and classify them as regular or irregular.

a) $x^2 y'' + y = 0$

b) $x(x^2-4)^2 y'' + (x^2+2x) y' + x^2 y = 0$

c) $y'' - (x+3)^{-2} y' + y/x = 0$

****

a) $y'' + \frac{1}{x^2} y = 0$ $x=0$ is a singular point

$x^2\left(\frac{1}{x^2}\right) = 1$ is analytic at $x=0$

So $x=0$ is a regular singular point

b) $y'' + \frac{x(x+2)}{x(x+2)^2(x-2)^2} y' + \frac{x^2}{x(x+2)^2(x-2)^2} y = 0$

or $y'' + \frac{1}{(x+2)(x-2)^2} y' + \frac{x}{(x+2)^2(x-2)^2} y = 0$

$x=-2$ and $x=2$ are singular points

$(x+2)\frac{1}{(x+2)(x-2)^2} = \frac{1}{(x-2)^2}$ is analytic at $x=-2$

$(x+2)^2\frac{x}{(x+2)^2(x-2)^2} = \frac{x}{(x-2)^2}$ is analytic at $x=-2$

So $x=-2$ is a regular singular point

$(x-2)\frac{1}{(x+2)(x-2)^2} = \frac{1}{(x+2)(x-2)}$ is not analytic at $x=2$

So $x=2$ is an irregular singular point

c) $y'' - \frac{1}{(x+3)^2} y' + \frac{1}{x} y = 0$

$x=-3$ and $x=0$ are singular points

a) $C_1 \sum_{n=0}^{\infty} a_n x^n + C_2 \sum_{n=0}^{\infty} b_n x^n$, b) $C_1 x^{2/3} \sum_{n=0}^{\infty} a_n x^n + C_2 x^{-2/3} \sum_{n=0}^{\infty} b_n x^n$

c) $C_1 x^{-2/3} \sum_{n=0}^{\infty} a_n x^n + C_2 x^{2/3} \ln x \sum_{n=0}^{\infty} b_n x^n$

Why ?

****

Write equation (1) in the form

$$y'' + P(x) y' + Q(x) y = 0$$

with $P(x) = \frac{e^x}{x(x+1)}$, $Q(x) = -\frac{4}{9} \frac{\cos x}{x^2(x+1)}$.

$P(x)$, $Q(x)$ goes to infinity as $x \to 0$, while $\lim_{x \to 0} x P(x)$ and $\lim_{x \to 0} x^2 Q(x)$ exist. Hence $x = 0$ is a regular-singular point of the equation (1). In order to determine the form of the solution, we first find the indicial equation

$$m(m-1) + p_0 m + q_0 = 0 \qquad (2)$$

where p_0 and q_0 are the first constant terms in the power series expansions of $xP(x)$ and $x^2 Q(x)$ respectively. In our case

$$xP(x) = \frac{e^x}{1+x} = \left(1 + x + \frac{x^2}{2} \cdots\right)\left(1 - x + x^2 + \cdots\right) = 1 + \cdots$$

$$x^2 Q(x) = -\frac{4}{9} \frac{\cos x}{1+x} = -\frac{4}{9}\left(1 + \frac{-x^2}{2} + \cdots\right)\left(1 - x + x^2 + \cdots\right) = -\frac{4}{9} + \cdots$$

Hence $p_0 = 1$, $q_0 = -\frac{4}{9}$ and (2) becomes

$$m^2 - \frac{4}{9} = 0$$

with two distinct roots $\frac{2}{3}$ and $-\frac{2}{3}$ not differing by an integer. Therefore the solution must take the form b).

3-14

Determine the singular points of the following equation and classify each singular point as regular or irregular.

$$x^2(x-3)y'' + (x+2)y' + 4y = 0.$$

* *

Dividing each term by $x^2(x-3)$, rewrite the equation as

$$y'' + \frac{x+2}{x^2(x-3)}y' + \frac{4}{x^2(x-3)}y = 0.$$

This equation is of the form

$$y'' + p(x)y' + q(x)y = 0$$

with

$$p(x) = \frac{x+2}{x^2(x-3)}, \quad q(x) = \frac{4}{x^2(x-3)}.$$

Since the functions $p(x)$ and $q(x)$ are singular at $x=0$ and at $x=3$, these points are the singular points of the differential equation.

Consider the point $x=0$ first. Note that

$$xp(x) = \frac{x+2}{x(x-3)}$$

is <u>not</u> analytic at $x=0$, although

$$x^2q(x) = \frac{4}{x-3}$$

$(x+3)\frac{1}{(x+3)^2} = \frac{1}{x+3}$ is not analytic at $x=-3$
so $x=-3$ is an irregular singular point

$(x)\frac{1}{(x+3)^2} = \frac{x}{(x+3)^2}$ is analytic at $x=0$

$(x^2)\frac{1}{x} = x$ is analytic at $x=0$
so $x=0$ is a regular singular point

3-13

Consider the differential equation

$$(4 - x^2)y'' - 2xy' + 6y = 0.$$

Where does it have singular points? Classify the singular points.

****

In standard form
$$y'' - \frac{2x}{4-x^2}y' + \frac{6}{4-x^2}y = 0.$$

Singular points occur where $x = \pm 2$.

$$\lim_{x\to 2}(x-2)\frac{2x}{x^2-4} = \lim_{x\to 2}\frac{2x(x-2)}{(x-2)(x+2)} = 1$$

$$\lim_{x\to 2}(x-2)^2\frac{6}{4-x^2} = \lim_{x\to 2}\frac{(x-2)^2 6}{(2-x)(2+x)} = 0$$

Thus $x=2$ is a regular singular point.
From the symmetry of the equation, $x=-2$ is also a regular singular point.

3-16

Classify each of the following differential equations as having an ordinary point, regular singular point or an irregular singular point at $x = 0$.

(a) $x^2y'' + (4x-1)\,y' + 2y = 0$

(b) $y'' + \frac{1}{x}\,y' + y = 0$

(c) $x(x+1)^2\,y'' + (1-x^2)\,y' - (1-x)\,y = 0$

(d) $(x+1)\,y'' + xy' + 3y = 0$

****

(a) $y'' + \frac{4x-1}{x^2}\,y' + \frac{2}{x^2}\,y = 0$. Since there is a zero in the denominator of the coefficients, these coefficients are not analytic - i.e. do not have power series expansions at $x=0$. So $x=0$ is not an ordinary point. We compute

$\lim_{x\to 0} x \cdot \frac{4x-1}{x^2} = \lim_{x\to 0} \frac{4x-1}{x}$ does not exist.

$\lim_{x\to 0} x^2 \cdot \frac{2}{x^2} = \lim_{x\to 0} 2 = 2$ does exist.

So $x=0$ is not a regular singular point. Hence $x=0$ is an irregular singular point.

(b) $y'' + \frac{1}{x}\,y' + y = 0$. $x=0$ is not an ordinary point (as above). We compute

$\lim_{x\to 0} x \cdot \frac{1}{x} = \lim_{x\to 0} 1 = 1$

$\lim_{x\to 0} x^2 \cdot 1 = 0$

Both limits exist, hence $x=0$ is a regular Singular point.

is analytic there. Consequently, $x=0$ is an irregular singular point.

Next examine the point $x = 3$. Since both the functions

$$(x-3)\,p(x) = \frac{x+2}{x^2}$$

and

$$(x-3)^2\,q(x) = \frac{4(x-3)}{x^2}$$

are analytic at $x = 3$, we conclude that $x = 3$ is a regular singular point.

3-15

List all of the singular points of $x^2(x^2-25)y'' + 9xy' - y = 0$

****

$$y'' + \frac{9x}{x^2(x^2-25)}y' - \frac{1}{x^2(x^2-25)}y = 0$$

$$y'' + \frac{9}{x(x^2-25)}y' - \frac{1}{x^2(x^2-25)}y = 0$$

Let $x^2(x^2-25) = 0$

$x = 0$, $x = \pm 5$ are the singular points.

3-17

Determine the singular points of the following differential equations and find the nature of their singularity:

i) $x^2y'' - xy' + 8y = 0$

ii) $(x^2-1)\,y'' - 2xy' + 4y = 0$

iii) $y'' + xy' + 5y = 0$

****

We put each equation into the form $y'' + P(x)y' + Q(x)y = 0$

i) $P(x) = \frac{-x}{x^2} = -\frac{1}{x}$, $Q(x) = \frac{8}{x^2}$.

At $x=0$, $P(x)$ and $Q(x)$ are not analytic.
$(x-0)P(x) = -1$ and $(x-0)^2Q(x) = x^2\frac{8}{x^2} = 8$ are both analytic at $x=0$.
Therefore $x=0$ is a regular singular point of the differential equation.

ii) $P(x) = -\frac{2x}{x^2-1}$, $Q(x) = \frac{4}{x^2-1}$

At $x=-1, 1$, $P(x)$ and $Q(x)$ are not analytic
$x=-1,1$ are singular points of the equation
At $x=-1$, $(x+1)P(x) = -\frac{2x}{x-1}$, $(x+1)^2Q(x) = \frac{4(x+1)}{x-1}$ are analytic at $x=-1$
At $x=1$, $(x-1)P(x) = -\frac{2x}{x+1}$, $(x-1)^2Q(x) = \frac{4(x-1)}{x+1}$ are analytic at $x=1$
Therefore $x=-1$ and $x=1$ are regular singular points.

iii) $P(x) = x$, $Q(x) = 5$
$P(x)$ and $Q(x)$ are analytic everywhere
Therefore every point $x = x_0$ is an ordinary point; that is the differential equation has no singular points.

(c) $y'' + \frac{1-x^2}{x(x+1)^2} y' - \frac{1-x}{x(x+1)^2} y = 0$. $x=0$ is not an ordinary point (as above). We compute

$$\lim_{x\to 0} x \cdot \frac{1-x^2}{x(x+1)^2} = \lim_{x\to 0} \frac{1-x^2}{(x+1)^2} = 1$$

$$\lim_{x\to 0} x^2 \cdot \left(-\frac{1-x}{x(x+1)^2}\right) = \lim_{x\to 0} \left[-\frac{x(1-x)}{(x+1)^2}\right] = 0$$

Both limits exist, hence $x=0$ is a regular singular point.

(d) $y'' + \frac{x}{x+1} y' + \frac{3}{x+1} y = 0$. At $x=0$ the coefficients of y' and y are analytic - i.e. have power series expansions (since the denominators are not 0 when $x=0$). Therefore $x=0$ is an ordinary point.

NOTE: In (a) we do not know the form of the solution (if any).

In (b) + (c) our computation shows that there are Frobenius Series solutions.

In (d) we conclude that there are power series solutions.

$y = \sum_{n=0}^{\infty} a_n x^n$; $y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$; $y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$.

$(2x^2+1)\sum_2^{\infty} n(n-1)a_n x^{n-2} + 3x\sum_1^{\infty} n a_n x^{n-1} - 3\sum_0^{\infty} a_n x^n = 0.$

$\sum_2^{\infty} 2n(n-1)a_n x^n + \sum_2^{\infty} n(n-1)a_n x^{n-2} + \sum_1^{\infty} 3n a_n x^n + \sum_0^{\infty} -3a_n x^n = 0.$

$\sum_2^{\infty} 2n(n-1)a_n x^n + \sum_0^{\infty} (n+2)(n+1)a_{n+2} x^n + \sum_1^{\infty} 3n a_n x^n + \sum_0^{\infty} -3a_n x^n = 0.$

$(2\cdot 1\cdot a_2 - 3a_0)x^0 + (3\cdot 2\cdot a_3 + 3\cdot 1\cdot a_1 - 3a_1)x +$

$+ \sum_2^{\infty} (2n(n-1)a_n + (n+2)(n+1)a_{n+2} + 3n a_n - 3a_n)x^n = 0.$

$a_2 = \frac{3}{2} a_0$; $a_3 = 0$. For $n \geq 2$, $a_{n+2} = -\frac{(2n+3)(n-1)a_n}{(n+2)(n+1)}$.

$a_4 = -\frac{7}{8} a_0$; $a_5 = 0$; $a_6 = \frac{77}{80} a_0$; $a_7 = 0$; $a_8 = -\frac{165}{128} a_0$; $\cdots$.

$y = a_0\left(1 + \frac{3}{2}x^2 - \frac{7}{8}x^4 + \frac{77}{80}x^6 - \frac{165}{128}x^8 + \cdots\right) + a_1 x$.

$y_1(x) = 1 + \sum_{n=1}^{\infty} \frac{(-1)^{n+1} 3\cdot 7\cdot 11 \cdots (4n-1)}{2^n (2n-1)\cdot n!} x^{2n}$;

$y_2(x) = x$.

3-20

Consider the equation $y'' - xy' - y = 0$. (1)

A) Find it's general solution in the form $a_0 y_1 + a_1 y_2$, where y_1 and y_2 are power series.

B) Use the ratio test to verify that the two series y_1 and y_2 converge for all x.

C) Show that y_1 is the series expansion of $e^{x^2/2}$.

****

A) Let $y = \sum_{n=0}^{\infty} a_n x^n$. Then $y' = \sum_{n=0}^{\infty} n a_n x^{n-1}$,

$y'' = \sum_{n=0}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$.

Make a change of index n in y'', i.e. let $n' = n-2$

3-18

Determine which of x=-2,x=0, and x=2 is a regular singular point of $x^2(x-2)y''+3y'-5(x-2)y=0$.

****

Rewrite the equation as $y'' + \frac{3}{x^2(x-2)}y' - \frac{5}{x^2}y = 0$

$x=-2$: ordinary point, since $\frac{3}{x^2(x-2)}$ and $\frac{-5}{x^2}$ are finite at $x=-2$.

$x=0$: singular point, not regular since $\lim_{x\to 0} x\,\frac{3}{x^2(x-2)}$ is

not finite (although $\lim_{x\to 0} x^2\,\frac{-5}{x^2}$ is finite)

$x=2$: regular singular point, since $\lim_{x\to 2}(x-2)\,\frac{3}{x^2(x-2)} = \frac{3}{4}$

and $\lim_{x\to 2}(x-2)^2\,\frac{-5}{x^2} = 0$.

SECOND ORDER WITH X = X_0 AN ORDINARY POINT

3-19

Determine two linearly independent Maclaurin series solutions of the differential equation $(2x^2 + 1)y'' + 3xy' - 3y = 0$. [Reminder: Maclaurin series = Taylor series (power series) with center of expansion $x_0 = 0$. Note that the point $x_0 = 0$ is an ordinary point for the differential equation.]

****

comparing with the series expansion of $e^3 = \sum_{n=0}^{\infty} \frac{1}{n!} 3^n$
we find that $y_1 = e^{\frac{x^2}{2}}$.

3-21

Find the general solution and the interval of convergence of the series solution $(1+2x^2)y'' - 5xy' + 3y = 0$

****

Find the singular points: Let $1+2x^2=0$ $\quad x^2 = -\frac{1}{2}$

The solution is valid if $|x| < \frac{\sqrt{2}}{2}$ $\quad x = \pm \frac{\sqrt{2}}{2} i$.

Solving about $x=0$, because $x=0$ is an ordinary point,
let $y = \sum_{n=0}^{\infty} a_n x^n$. Substitute into the original equation.

$$\sum_{n=0}^{\infty} n(n-1)a_n x^{n-2} + \sum_{n=0}^{\infty} 2n(n-1)a_n x^n + \sum_{n=0}^{\infty} -5na_n x^n + \sum_{n=0}^{\infty} 3a_n x^n = 0$$

$$\sum_{n=0}^{\infty} n(n-1)a_n x^{n-2} + \sum_{n=2}^{\infty} \left[2(n-2)(n-3) - 5(n-2) + 3\right] a_{n-2} x^{n-2} = 0$$

a_0 and a_1 are arbitrary constants).

If $n \geq 2$: $\quad a_n = -\frac{(2n^2-15n+25)a_{n-2}}{n(n-1)} = -\frac{(2n-5)(n-5)a_{n-2}}{n(n-1)}$

So $\quad a_3 = -\frac{(-2)(1)}{(2)(3)} a_1 = \frac{1}{3} a_1$

$a_5 = 0a_3 = 0$

$\therefore a_{2k+1} = 0$ if $k \geq 2$

then $y'' = \sum_{n'=0}^{\infty} (n'+2)(n'+1)\, a_{n'+2} x^{n'} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n$.

Substitute the above expressions into (1):

$$\sum_{n=0}^{\infty} \left[(n+2)(n+1)\, a_{n+2} - n a_n - a_n \right] x^n = 0$$

Hence $(n+2)(n+1)a_{n+2} - n a_n - a_n = 0$

i.e. $$a_{n+2} = \frac{1}{n+2} a_n \qquad (2)$$

Use this recursion formula to find all the coefficients.

i) For coefficients with even index,

$$a_{2n} = \frac{1}{2n} a_{2n-2} = \frac{1}{2n(2n-2)} a_{2n-4} = \cdots = \frac{1}{2n(2n-2)\cdots 4\cdot 2} a_0$$

ii) For coefficients with odd index

$$a_{2n+1} = \frac{1}{2n+1} a_{2n-1} = \frac{1}{(2n+1)(2n-1)} a_{2n-3} = \cdots = \frac{1}{(2n+1)(2n-1)\cdots 5\cdot 3\cdot 1} a_1$$

Let $y_1 = \sum_{n=1}^{\infty} \frac{x^{2n}}{2n(2n-2)\cdots 4\cdot 2} + 1$, $y_2 = \sum_{n=0}^{\infty} \frac{1}{(2n+1)\cdots 5\cdot 3\cdot 1} a_1$

Then the general solution of (1) is $a_0 y_1 + a_1 y_2$.

B) Either in the power series y_1 or y_2, the two successive coefficients are a_n and a_{n+2}. By (2)

$$\left| \frac{a_{n+2}}{a_n} \right| = \frac{1}{n+2} \to 0 \quad \text{as } n \to \infty.$$

Hence by the ratio test for power series, the series y_1 and y_2 converge for all x.

C) Writing y_1 as $\sum_{n=0}^{\infty} \frac{x^{2n}}{2^n n!} = \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{x^2}{2}\right)^n$ and

$$y'(x) = \quad C_1 \quad + 2C_2x + \_\_\_\_\_ + kC_kx^{k-1} + \_\_\_\_\_$$

Similarly,

$$y''(x) = \quad 2C_2 \quad + \_\_\_\_\_ + (k-1)kC_kx^{k-2} + \_\_\_\_$$

Therefore,

$$x^2y'(x) = \quad C_1x^2 \quad + 2C_2x^3 + \_\_\_\_ + kC_kx^{k+1} + \_\_\_\_\_$$

and

$$xy(x) = C_0x + C_1x^2 + C_2x^3 + \_\_\_\_ + C_kx^{k+1} + \_\_\_\_$$

We observe that the coefficient of x^k in the expansion of

$y''(x)$ is $(k+1)(k+2)C_{k+2}$,

The coefficient of x^k in the expansion of $x^2y'(x)$ is

$(k-1)C_{k-1}$

and the coefficient of x^k in the expansion of $xy(x)$ is

C_{k-1}.

Thus, the coefficient of x^k in the expansion of

$y'' + x^2y' + xy$ is

$$(k+1)(k+2)C_{k+2} + (k-1)C_{k-1} + C_{k-1}.$$

Since $y'' + x^2y' + xy$ is zero, the coefficient of each x^k is zero that is

$$(k+1)(k+2)C_{k+2} + kC_{k-1} = 0.$$

We obtain

$$C_{k+2} = \frac{-k}{(k+1)(k+2)}C_{k-1}$$

$$a_2 = -\frac{(-1)(-3)}{(1)(2)}\,a_0$$

$$a_4 = -\frac{(3)(-1)}{(3)(4)}\,a_2$$

$$\vdots$$

$$a_{2K} = -\frac{(4K-5)(2K-5)}{(2K-1)(2K)}\,a_{2K-2} \qquad \text{for } K \geq 1.$$

Multiplying: $$a_{2K} = \frac{(-1)^K\left[(-1)(3)(7)\cdots(4K-5)\right]\left[(-3)(-1)\cdots(2K-5)\right]a_0}{\left[(1)(3)(5)\cdots(2K-1)\right]\left[(2)(4)\cdots(2K)\right]}$$

$$= \frac{(-1)^K\left[(-1)(3)(7)\cdots(4K-5)\right](3)\,a_0}{(2K-3)(2K-1)\,2^K\cdot K!}$$

$$\therefore\quad y = a_0\left[1 + \sum_{K=1}^{\infty}\frac{(-1)^K\left[(-1)(3)\cdots(4K-5)\right]3X^{2K}}{(2K-3)(2K-1)\,2^K\cdot K!}\right] + a_1\left(X + \frac{1}{3}X^3\right)$$

3-22

$y'' + x^2y' + xy = 0$. Let $y = \sum_k c_k x^k$, where c_k are constants and let c_0 and c_1 be given.

$c_3 =$ (A) $-c_0/6$ (B) $-c_1/6$ (C) c_0 (D) c_1 (E) $c_1/6$.

****

Since $y(x)$ has a power series expansion

$$y(x) = C_0 + C_1x + C_2x^2 + \cdots\cdots + C_kx^k + \cdots,$$

its derivative may be obtained applying term by term differentiation. That is

$$\underbrace{(2a_2 + a_0)}_{x^0 \text{ term}} + \sum_{k=1}^{\infty} \left[(k+2)(k+1)a_{k+2} + (k+1)a_k\right] x^k = 0$$

Equate coefficients on the left to the corresponding coefficient on the right (remember: $0 = \sum_{k=1}^{\infty} 0 \cdot x^k$).

$$2a_2 + a_0 = 0 \qquad a_2 = -\tfrac{1}{2}a_0$$

$$(k+2)(k+1)a_{k+2} + (k+1)a_k = 0 \qquad a_{k+2} = -\frac{1}{k+2}a_k$$
$$k = 1, 2, 3, \cdots \qquad k = 1, 2, 3, \cdots$$

So a_0 = arbitrary
a_1 = arbitrary

$$a_2 = -\tfrac{1}{2}a_0$$

$k=1$: $a_3 = -\frac{1}{3}a_1$

$k=2$: $a_4 = -\frac{1}{4}a_2 = -\frac{1}{4}(-\frac{1}{2}a_0) = \frac{1}{2\cdot 4}a_0$

$k=3$: $a_5 = -\frac{1}{5}a_3 = -\frac{1}{5}(-\frac{1}{3}a_1) = \frac{1}{3\cdot 5}a_1$

etc.

Putting these into $y = \sum_{n=0}^{\infty} a_n x^n$ gives

$$y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + \cdots$$

$$y = a_0 + a_1 x - \tfrac{1}{2}a_0 x^2 - \tfrac{1}{3}a_1 x^3 + \frac{1}{2\cdot 4}a_0 x^4 + \frac{1}{3\cdot 5}a_1 x^5 + \cdots$$

$$y = a_0\left(1 - \tfrac{1}{2}x^2 + \frac{1}{2\cdot 4}x^4 + \cdots\right) + a_1\left(x - \tfrac{1}{3}x^3 + \frac{1}{3\cdot 5}x^5 + \cdots\right)$$

In particular, for $k=1$

$$C_3 = C_{1+2} = \frac{-1}{(1+1)(1+2)} C_{1-1} = \frac{-1}{6} C_0.$$

The correct answer is (A).

used

3-23

Find the first 6 nonzero terms in the power series solution to $y'' + xy' + y = 0$.

****

$$y = \sum_{n=0}^{\infty} a_n x^n, \quad y' = \sum_{n=1}^{\infty} n a_n x^{n-1}, \quad y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

$y'' + xy' + y = 0$ becomes

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + x \sum_{n=1}^{\infty} n a_n x^{n-1} + \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + \sum_{n=1}^{\infty} n a_n x^n + \sum_{n=0}^{\infty} a_n x^n = 0$$

Let $n = k+2$ $(k = n-2)$ in the 1st term and $k = n$ in the 2nd + 3rd terms.

$$\sum_{k=0}^{\infty} (k+2)(k+1) a_{k+2} x^k + \sum_{k=1}^{\infty} k a_k x^k + \sum_{k=0}^{\infty} a_k x^k = 0$$

$$(2)(1) a_2 + \sum_{k=1}^{\infty} (k+2)(k+1) a_{k+2} x^k + \sum_{k=1}^{\infty} k a_k x^k + a_0 + \sum_{k=1}^{\infty} a_k x^k = 0$$

used

3-25

Find a general solution to $(2x+1)y'' + 4xy' - 4y = 0$ about the ordinary point $x = 0$.

****

The equation has two linearly independent solutions of the form $y = \sum_{n=0}^{\infty} a_n x^n$. Substituting in the D.E.,

$$(2x+1)\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + 4x\sum_{n=1}^{\infty} n a_n x^{n-1} - 4\sum_{n=0}^{\infty} a_n x^n = 0,$$

$$\text{i.e. } \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + \sum_{n=2}^{\infty} 2n(n-1)a_n x^{n-1} + \sum_{n=1}^{\infty} 4n a_n x^n - \sum_{n=0}^{\infty} 4a_n x^n = 0.$$

Replace n by $k+2$ in the first sum, $k+1$ in the second sum and k in the last two sums to obtain

$$\sum_{k=0}^{\infty}(k+2)(k+1)a_{k+2}x^k + \sum_{k=1}^{\infty} 2(k+1)k\,a_{k+1}x^k + \sum_{k=1}^{\infty} 4k a_k x^k - \sum_{k=0}^{\infty} 4a_k x^k = 0,$$

$$\text{i.e. } 2a_2 - 4a_0 + \sum_{k=1}^{\infty}\left[(k+2)(k+1)a_{k+2} + 2(k+1)k\,a_{k+1} + 4(k-1)a_k\right]x^k = 0.$$

This leads us to

$$2a_2 - 4a_0 = 0 \quad , \text{ i.e. } \quad a_2 = 2a_0 ,$$

$$(k+2)(k+1)a_{k+2} = -\left[2(k+1)k\,a_{k+1} + 4(k-1)a_k\right], \quad k = 1, 2, \ldots$$

$$k=1: \quad (3)(2)a_3 = -[4a_2 + 0]$$

$$a_3 = -\frac{2}{3}a_2 = -\frac{4}{3}a_0 = -\frac{2^3}{3.2}a_0 = -\frac{2^3}{3!}a_0.$$

$$k=2: \quad (4)(3)a_4 = -[12a_3 + 4a_2]$$

$$a_4 = -\frac{[12a_3 - 6a_3]}{4.3} = -\frac{2a_3}{4} = \frac{2^4}{4!}a_0$$

It follows that $a_k = \frac{(-1)^k 2^k}{k!}a_0$, $k = 2, 3, \ldots$, a_0 and a_1 arbitrary. If we write $a_1 = b - a_0$ and $a = a_0$, then

$$y = \sum_{k=0}^{\infty} \frac{(-1)^k 2^k}{k!} a x^k + bx = ae^{-2x} + bx$$

Thus, our linearly independent solutions are

$$y = e^{-2x} \quad \text{and} \quad y = x.$$

3-24

Find the general solution of $y'' - 4xy' = 0$. used

****

$P(x) = -4x$, $Q(x) = 0$ are analytic function. The diff. equation has no singular points.

Assume $y = \sum_{n=0}^{\infty} a_n x^n$, $\frac{dy}{dx} = \sum_{n=0}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n$

$\frac{d^2y}{dx^2} = \sum_{n=0}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n$

Substituting the series for $\frac{d^2y}{dx^2}$ and $\frac{dy}{dx}$ in the given diff. equation, we obtain

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n - 4x \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n = 0$$

Setting the coefficient of x^n equal to zero gives

$$a_2 = 0$$

$$(n+2)(n+1) a_{n+2} - 4n a_n = 0$$

$$\text{Therefore } a_{n+2} = \frac{4n}{(n+1)(n+2)} a_n$$

$a_2 = 0$ results in $a_4 = a_6 = a_8 = \cdots\cdots = 0$

$$a_3 = \frac{4}{(2)(3)} a_1 = \frac{2}{3} a_1, \quad a_5 = \frac{4(3)}{(4)(5)} a_3 = \frac{3}{5} \cdot \frac{2}{3} a_1 = \frac{2}{5} a_1$$

$$a_7 = \frac{4(5)}{(6)(7)} a_5 = \frac{4(5)}{(6)(7)} \frac{2}{5} a_1 = \frac{4}{21} a_1$$

$$a_9 = \frac{2}{27} a_1, \quad a_{11} = \frac{4}{165} a_1$$

Therefore the general solution is

$$y(x) = a_0 + a_1 \left(x + \frac{2}{3}x^3 + \frac{2}{5}x^5 + \frac{4}{21}x^7 + \frac{2}{27}x^9 + \cdots\right)$$

$$= a_0 + a_1 \sum_{k=1}^{\infty} \frac{2^k}{k!\,(2k+1)} x^{2k+1}$$

3-27

Solve: $y'' - 2xy = 0$.

****

AN INFINITE SERIES SOLUTION IS REQUIRED. $X=0$ IS AN ORDINARY POINT SO A SIMPLE POWER SERIES WILL YIELD A GENERAL SOLUTION.

LET $y = \sum_{n=0}^{\infty} a_n x^n$. THEN $y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$, $y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$.

SUBSTITUTING INTO $y'' - 2xy$, AND NOTING THAT $x\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^{n+1}$,

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} - \sum_{n=0}^{\infty} 2a_n x^{n+1}$$ RESULTS. WE REINDEX THE 2ND SUM:

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} - \sum_{n=3}^{\infty} 2a_{n-3} x^{n-2}$$, AND WRITE THE $n=2$ TERM BY ITSELF:

$$2a_2 + \sum_{n=3}^{\infty} n(n-1) a_n x^{n-2} - \sum_{n=3}^{\infty} 2a_{n-3} x^{n-2} = 2a_2 + \sum_{n=3}^{\infty} \left[a_n (n)(n-1) - 2a_{n-3}\right] x^{n-2}.$$

IF $y'' - 2xy = 0$, THEN $a_2 = 0$ AND $a_n (n)(n-1) - 2a_{n-3} = 0$, $n = 3, 4, \ldots$.

$\therefore\ a_n = \dfrac{2}{n(n-1)} a_{n-3}$ AND THE RELATION SUGGESTS 3 COLUMNS.

$a_3 = \dfrac{2}{3\cdot 2} a_0$

$a_6 = \dfrac{2}{6\cdot 5} a_3 = \dfrac{2^2}{6\cdot 5\cdot 3\cdot 2} a_0$

$a_9 = \dfrac{2}{9\cdot 8} a_6 = \dfrac{2^3}{9\cdot 8\cdot 6\cdot 5\cdot 3\cdot 2} a_0$

...

$a_4 = \dfrac{2}{4\cdot 3} a_1$

$a_7 = \dfrac{2}{7\cdot 6} a_4 = \dfrac{2^2}{7\cdot 6\cdot 4\cdot 3} a_1$

$a_{10} = \dfrac{2}{10\cdot 9} a_7 = \dfrac{2^3}{10\cdot 9\cdot 7\cdot 6\cdot 4\cdot 3} a_1$

...

$a_5 = \dfrac{2}{5\cdot 4} a_2 = 0.$

$a_8 = \dfrac{2}{8\cdot 7} a_5 = 0.$

$a_{11} = \dfrac{2}{11\cdot 10} a_8 = 0.$

IN FACT, WE CAN CONCLUDE THAT

$a_{3K+2} = 0$, $K = 1, 2, 3, \ldots$.

used

3-26

Find the first five nonzero terms in a power series expansion about $x = 0$ for solution to the initial value problem

$$y'' + y' - xy = 0; \quad y(0) = -1, \quad y'(0) = 2$$

****

Since $x = 0$ is an ordinary point, we assume that

$y = \sum_{n=0}^{\infty} c_n x^n$. So $y' = \sum_{n=1}^{\infty} n c_n x^{n-1}$ and $y'' = \sum_{n=2}^{\infty} n(n-1) c_n x^{n-2}$

Substituting,

$$\sum_{n=2}^{\infty} n(n-1) c_n x^{n-2} + \sum_{n=1}^{\infty} n c_n x^{n-1} - \sum_{n=0}^{\infty} c_n x^{n+1} = 0.$$

(Let $K = n-2$) (Let $K = n-1$) (Let $K = n+1$)

Shifting indices,

$$\sum_{K=0}^{\infty} (K+2)(K+1) c_{K+2} x^K + \sum_{K=0}^{\infty} (K+1) c_{K+1} x^K - \sum_{K=1}^{\infty} c_{K-1} x^K = 0$$

Simplifying,

$$2c_2 + c_1 + \sum_{K=1}^{\infty} [(K+2)(K+1) c_{K+2} + (K+1) c_{K+1} - c_{K-1}] x^K = 0.$$

So we have $c_2 = \frac{-c_1}{2}$ and $c_{K+2} = \frac{c_{K-1} - (K+1) c_{K+1}}{(K+2)(K+1)}$

Now, the initial conditions imply that $c_0 = -1$ and $c_1 = 2$.

Thus, we have $c_2 = -1$, $c_3 = 1/6$ and $c_4 = 1/8$.

and the first five terms of the solution are

$$y = -1 + 2x - x^2 + \frac{x^3}{6} + \frac{x^4}{8}$$

3-28

Find the recurrence relation for the power series solution to:

$$(1 - x^2)y'' - 4xy' - 2y = 0; \quad x_0 = 0.$$

****

$$y = \sum_{n=0}^{\infty} a_n x^n \qquad \text{since} \quad x_0 = 0.$$

$$y' = \sum_{n=0 \text{ or } 1}^{\infty} n a_n x^{n-1}, \text{ and} \quad y'' = \sum_{n=0,1,\text{or } 2}^{\infty} n(n-1) a_n x^{n-2}$$

(The y' series can be started at 1 since the $n=0$ term is $\equiv 0$. Ditto the y'' series.)

$$(1-x^2)y'' - 4xy' - 2y = 0 \qquad \text{gives}$$

$$\sum_{n=0,1,\text{or } 2}^{\infty} n(n-1) a_n x^{n-2} - \sum_{n=0,1,\text{or } 2}^{\infty} n(n-1) a_n x^n$$

$$- 4 \sum_{n=0 \text{ or } 1}^{\infty} n a_n x^n - 2 \sum_{n=0}^{\infty} a_n x^n = 0,$$

or

$$\sum_{n=-2,-1,\text{or } 0}^{\infty} (n+2)(n+1) a_{n+2} x^n - \sum_{n=0,1,\text{or } 2}^{\infty} n(n-1) a_n x^n$$

$$- 4 \sum_{n=0 \text{ or } 1}^{\infty} n a_n x^n - 2 \sum_{n=0}^{\infty} a_n x^n = 0,$$

or

$$\sum_{n=0}^{\infty} \left[(n+2)(n+1) a_{n+2} - n(n-1) a_n - 4n a_n - 2a_n \right] x^n = 0$$

so

$$(n+2)(n+1) a_{n+2} - n(n-1) a_n - 4n a_n - 2a_n = 0$$

and

$$(n+2)(n+1) a_{n+2} = (n^2 - n + 4n + 2) a_n$$

$$(n^2 + 3n + 2) a_{n+2} = (n^2 + 3n + 2) a_n = (n+2)(n+1) a_n$$

$$\underline{\underline{a_{n+2} = a_n}}.$$

IN GENERAL,

$$a_{3K} = \frac{2^K}{3K(3K-1)(3K-3)(3K-4)\cdots 6\cdot 5\cdot 3\cdot 2}\, a_0, \quad K = 1, 2, 3, \cdots,$$

$$= \frac{2^K (3K-2)(3K-5)\cdots 7\cdot 4\cdot 1}{(3K)!}\, a_0, \quad K = 1, 2, 3, \cdots.$$

$$a_{3K+1} = \frac{2^K}{(3K+1)(3K)(3K-2)(3K-3)\cdots 7\cdot 6\cdot 4\cdot 3}\, a_1, \quad K = 1, 2, 3, \cdots,$$

$$= \frac{2^K (3K-1)(3K-4)\cdots 8\cdot 5\cdot 2}{(3K+1)!}\, a_1, \quad K = 1, 2, 3, \cdots.$$

Now $\sum_{m=0}^{\infty} a_m x^m = a_0 + a_1 x + a_2 x^2 + \sum_{K=1}^{\infty} a_{3K} x^{3K} + \sum_{K=1}^{\infty} a_{3K+1} x^{3K+1} + \sum_{K=1}^{\infty} a_{3K+2} x^{3K+2}$ IS AN IDENTITY.

$$\therefore y(x) = a_0 + a_1 x + 0 + \sum_{K=1}^{\infty} \frac{2^K (3K-2)(3K-5)\cdots 7\cdot 4\cdot 1}{(3K)!}\, a_0 x^{3K} + \sum_{K=1}^{\infty} \frac{2^K (3K-1)(3K-4)\cdots 8\cdot 5\cdot 2}{(3K+1)!}\, a_1 x^{3K+1}$$

$$= a_0 \left[1 + \sum_{K=1}^{\infty} \frac{2^K (3K-2)(3K-5)\cdots 7\cdot 4\cdot 1}{(3K)!}\, x^{3K}\right] + a_1 \left[x + \sum_{K=1}^{\infty} \frac{2^K (3K-1)(3K-4)\cdots 8\cdot 5\cdot 2}{(3K+1)!}\, x^{3K+1}\right].$$

THESE SERIES ARE LINEARLY INDEPENDENT, AND a_0 AND a_1 ARE ARBITRARY, SO THIS IS A GENERAL SOLUTION. THE SERIES CONVERGE FOR ALL FINITE x.

3-30

Write out at least 4 terms for each of the linearly independent solutions of $y''+(x-1)^2y'+(x-2)y=0$ in powers of $x-1$, and find the values of the arbitrary constants which make $y(1)=1$ and $y'(1)=-2$.

****

With $u=x-1$, the problem becomes $y''+u^2y'+(u-1)y=0$

where y is a function of u and y' now denotes dy/du.

Assume $y=\sum_{n=0}^{\infty}a_nu^n$, then $y'=\sum_{n=1}^{\infty}na_nu^{n-1}$, $y''=\sum_{n=2}^{\infty}n(n-1)a_nu^{n-2}$.

$$y''+u^2y'+(u-1)y=\sum_{n=2}^{\infty}n(n-1)a_nu^{n-2}+\sum_{n=1}^{\infty}u^2na_nu^{n-1}+\sum_{n=0}^{\infty}(u-1)a_nu^n=0$$

We want all sums to have the common power u^m :

$$\sum_{n=2}^{\infty}n(n-1)a_nu^{n-2}=\sum_{m=0}^{\infty}(m+2)(m+1)a_{m+2}u^m \qquad (m=n-2)$$

$$\sum_{n=1}^{\infty}u^2na_nu^{n-1}=\sum_{n=1}^{\infty}na_nu^{n+1}=\sum_{m=2}^{\infty}(m-1)a_{m-1}u^m \qquad (m=n+1)$$

$$\sum_{n=0}^{\infty}(u-1)a_nu^n=\sum_{n=0}^{\infty}a_nu^{n+1}-\sum_{n=0}^{\infty}a_nu^n=\sum_{m=1}^{\infty}a_{m-1}u^m-\sum_{m=0}^{\infty}a_mu^m$$

Then

$$\sum_{m=0}^{\infty}(m+2)(m+1)a_{m+2}u^m+\sum_{m=2}^{\infty}(m-1)a_{m-1}u^m+\sum_{m=1}^{\infty}a_{m-1}u^m-\sum_{m=0}^{\infty}a_mu^m=0$$

$$2a_2+6a_3u+a_0u-a_0-a_1u+\sum_{m=2}^{\infty}\left[(m+1)(m+2)a_{m+2}+(m-1)a_{m-1}+a_{m-1}-a_m\right]\times u^m=0$$

Therefore,

$$2a_2-a_0=0 \quad\Rightarrow\quad a_2=\tfrac{1}{2}a_0$$

$$6a_3+a_0-a_1=0 \quad\Rightarrow\quad a_3=\tfrac{1}{6}a_1-\tfrac{1}{6}a_0$$

3-29

Find a power series solution of $y'' + xy' - y = e^x$.

****

Let $y = \sum_{n=0}^{\infty} a_n x^n$. Then

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}, \quad y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}.$$

Substituting these in the equation, we obtain

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + \sum_{n=1}^{\infty} n a_n x^n - \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

Replacing n by k+2 in the first sum and by k in the other sums, we have

$$\sum_{k=0}^{\infty} (k+2)(k+1) a_{k+2} x^k + \sum_{k=1}^{\infty} k a_k x^k - \sum_{k=0}^{\infty} a_k x^k = \sum_{k=0}^{\infty} \frac{x^k}{k!},$$

i.e. $$(2a_2 - a_0) + \sum_{k=1}^{\infty} \left[(k+2)(k+1) a_{k+2} + (k-1) a_k\right] = 1 + \sum_{k=1}^{\infty} \frac{x^k}{k!}$$

$\therefore \quad 2a_2 - a_0 = 1$ and

$$(k+2)(k+1) a_{k+2} = \frac{1}{k!} - (k-1) a_k, \quad k = 1, 2, \ldots$$

$k=1$: $(3)(2) a_3 = 1$, i.e. $a_3 = \frac{1}{3!}$

$k=2$: $(4)(3) a_4 = \frac{1}{2!} - a_2 = \frac{1}{2!} - \frac{(1+a_0)}{2} = \frac{-a_0}{2}$,

i.e. $a_4 = \frac{-a_0}{4!}$

$k=3$: $(5)(4) a_5 = \frac{1}{3!} - 2a_3 = \frac{1}{3!} - \frac{2}{3!} = -\frac{1}{3!}$,

i.e. $a_5 = -\frac{1}{5!}$

Thus

$$y(x) = a_0 + a_1 x + \frac{(1+a_0)}{2!} x^2 + \frac{1}{3!} x^3 + \frac{-a_0}{4!} x^4 + \frac{-1}{5!} x^5 + \ldots$$

where a_0 and a_1 are arbitrary.

Assume $y=\sum_{n=0}^{\infty}c_n t^n$, $y'=\sum_{n=1}^{\infty}n c_n t^{n-1}$

$$y''=\sum_{n=2}^{\infty}n(n-1)c_n t^{n-2}$$

Substitute into the DE

$$0=(t+1)y''(t)-ty'(t)+y(t)$$

$$=(t+1)\sum_{n=2}^{\infty}n(n-1)c_n t^{n-2}-t\sum_{n=1}^{\infty}n c_n t^{n-1}+\sum_{n=0}^{\infty}c_n t^n$$

$$=\sum_{n=2}^{\infty}n(n-1)c_n t^{n-1}+\sum_{n=2}^{\infty}n(n-1)c_n t^{n-2}-\sum_{n=1}^{\infty}n c_n t^n+\sum_{n=0}^{\infty}c_n t^n$$

Write all summations so that t^n appears.

$$0=\sum_{n=1}^{\infty}(n+1)n c_{n+1}t^n+\sum_{n=0}^{\infty}(n+2)(n+1)c_{n+2}t^n-\sum_{n=1}^{\infty}n c_n t^n+\sum_{n=0}^{\infty}c_n t^n$$

$$=2c_2+c_0+\sum_{n=1}^{\infty}\left[n(n+1)c_{n+1}+(n+2)(n+1)c_{n+2}+(1-n)c_n\right]t^n$$

From the initial conditions $c_0=1$, $c_1=-1$.

From the power series solution, $c_2=-\frac{1}{2}c_0=-\frac{1}{2}$

Recurrence formula

$$\boxed{n(n+1)c_{n+1}+(n+2)(n+1)c_{n+2}+(1-n)c_n=0,\ n\geq 1}$$

Recurrence relation: $(m+1)(m+2)a_{m+2} + (m-1)a_{m-1} + a_{m-1} - a_m = 0$

or $a_{m+2} = \dfrac{a_m - m a_{m-1}}{(m+1)(m+2)}$ $\quad m = 2, 3, \ldots$

$m=2$: $a_4 = \dfrac{a_2 - 2a_1}{12} = \dfrac{\frac{1}{2}a_0 - 2a_1}{12} = \frac{1}{24}a_0 - \frac{1}{6}a_1$

$m=3$: $a_5 = \dfrac{a_3 - 3a_2}{20} = \dfrac{\frac{1}{6}a_1 - \frac{1}{6}a_0 - \frac{3}{2}a_0}{20} = \frac{1}{120}a_1 - \frac{1}{12}a_0$

We now have 5 terms in a_0 and 4 terms in a_1, so we stop.

$y(u) = a_0 + a_1 u + a_2 u^2 + \ldots = a_0 + a_1 u + (\frac{1}{2}a_0)u^2 + (\frac{1}{6}a_1 - \frac{1}{6}a_0)u^3 + \ldots$

$y(u) = a_0\left[1 + \frac{1}{2}u^2 - \frac{1}{6}u^3 + \frac{1}{24}u^4 - \frac{1}{12}u^5 + \ldots\right] + a_1\left[u + \frac{1}{6}u^3 - \frac{1}{6}u^4 + \frac{1}{120}u^5 \ldots\right]$

$y(x) = a_0\left[1 + \frac{1}{2}(x-1)^2 - \frac{1}{6}(x-1)^3 + \frac{1}{24}(x-1)^4 - \frac{1}{12}(x-1)^5 + \ldots\right] + a_1\left[(x-1) + \frac{1}{6}(x-1)^3 - \frac{1}{6}(x-1)^4 + \frac{1}{120}(x-1)^5 \ldots\right]$

With $x=1$, $y(1) = a_0$ and $y'(1) = a_1$ so to satisfy the initial conditions simply set $a_0 = 1$ and $a_1 = -2$.

3-31

Find the first five terms (to the 4th power) of the series solution about x = 1 to the IVP

$$xy'' + (1 - x)y' + y = 0$$

$$y(1) = 1,\ y'(1) = -1.$$

State explicitly the recurrence formula.

****

Change variables so that $t = x - 1$. Then

$(t+1)y''(t) - ty'(t) + y(t) = 0$.

The differential equation becomes

$$\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r} + \sum_{n=0}^{\infty}(n+r)a_n x^{n+r} - \sum_{n=1}^{\infty}(n+r-1)a_{n-1}x^{n+r} + \sum_{n=0}^{\infty} -2a_n x^{n+r} = 0$$

$$\left[r(r-1)+r-2\right]a_0 x^r + \sum_{n=1}^{\infty}\left\{\left[(n+r)(n+r-1)+(n+r)-2\right]a_n - (n+r-1)a_{n-1}\right\}x^{n+r} = 0$$

Indicial equation : $r(r-1)+r-2 = r^2-2 = 0$; $r = \pm\sqrt{2}$

For $r=\sqrt{2}$, the recurrence relation is

$$\left[n^2+2\sqrt{2}\,n\right]a_n = (n+\sqrt{2}-1)a_{n-1}, \text{ or } a_n = \frac{n+\sqrt{2}-1}{n^2+2\sqrt{2}n}a_{n-1}$$

$$n=1: \quad a_1 = \frac{\sqrt{2}}{1+2\sqrt{2}}a_0$$

$$n=2: \quad a_2 = \frac{1+\sqrt{2}}{4+4\sqrt{2}}a_1 = \frac{1}{4}a_1 = \frac{\sqrt{2}}{4+8\sqrt{2}}a_0$$

$$y_1(x) = a_0\left[1+\frac{\sqrt{2}}{1+2\sqrt{2}}x + \frac{\sqrt{2}}{4+8\sqrt{2}}x^2 + \cdots\right]$$

For $r=-\sqrt{2}$, the recurrence relation is

$$\left[n^2-2\sqrt{2}\,n\right]b_n = \left[n-\sqrt{2}-1\right]b_{n-1} \text{ or } b_n = \frac{n-\sqrt{2}-1}{n^2-2\sqrt{2}\,n}b_{n-1}$$

$$n=1: \quad b_1 = \frac{-\sqrt{2}}{1-2\sqrt{2}}b_0$$

$$n=2: \quad b_2 = \frac{1-\sqrt{2}}{4-4\sqrt{2}}b_1 = \frac{1}{4}b_1 = \frac{-\sqrt{2}}{4-8\sqrt{2}}b_0$$

$$y_2(x) = b_0\left[1-\frac{\sqrt{2}}{1-2\sqrt{2}}x - \frac{\sqrt{2}}{4-8\sqrt{2}}x^2 + \cdots\right]$$

$$c_{n+2} = \frac{(n-1)c_n - n(n+1)c_{n+1}}{(n+2)(n+1)}$$

$n=1$: $c_3 = \frac{0\cdot c_1 - 2c_2}{3\cdot 2} = \frac{1}{6}$

$n=2$: $c_4 = \frac{(2-1)c_2 - 2(3)c_3}{4\cdot 3} = \frac{-\frac{1}{2}-1}{12} = -\frac{1}{8}$

Hence $y(t) = c_0 + c_1 t + c_2 t^2 + c_3 t^3 + c_4 t^4 + \cdots$

$$y(x) = 1 - (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{6}(x-1)^3 - \frac{1}{8}(x-1)^4 + \cdots$$

SECOND ORDER WITH X = X_0 A REGULAR SINGULAR POINT (FROBENIUS SERIES)

DISTINCT ROOTS NOT DIFFERING BY AN INTEGER

3-32

For $x^2y'' + (x-x^2)y'-2y=0$, find 3 terms in powers of x for each of the independent solutions.

****

If $y = \sum_{n=0}^{\infty} a_n x^{n+r}$, $y' = \sum_{n=0}^{\infty} (n+r)a_n x^{n+r-1}$, $y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1)a_n x^{n+r-2}$

Then $x^2y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1)a_n x^{n+r}$,

$$(x-x^2)y' = \sum_{n=0}^{\infty} (n+r)a_n x^{n+r} - \sum_{n=0}^{\infty} (n+r)a_n x^{n+r+1}$$

$$= \sum_{n=0}^{\infty} (n+r)a_n x^{n+r} - \sum_{n=1}^{\infty} (n+r-1)a_{n-1} x^{n+r}$$

3-34

Find the general solution of

$$8x^2y'' + 2xy' + (x+1)y = 0 \quad \text{near } x = 0.$$

****

$P(x) = \frac{2x}{8x^2} = \frac{1}{4x}$ and $Q(x) = \frac{x+1}{x^2} = \frac{1}{x} + \frac{1}{x^2}$

$P(x)$ and $Q(x)$ are not analytic at $x=0$, that is $x=0$ is a singular point of the given differential equation.

$(x-0)P(x) = x\,\frac{1}{4x} = \frac{1}{4}$ and $(x-0)^2 Q(x) = x^2\left(\frac{x+1}{x^2}\right) = x+1$

$(x-0)P(x)$ and $(x-0)^2Q(x)$ are analytic at $x=0$ Therefore $x=0$ is a regular singular point of the equation.

Assume $y(x) = x^k \sum_{n=0}^{\infty} a_n x^n$ Frobenius Series

$$= \sum_{n=0}^{\infty} a_n x^{n+k}$$

$$y'(x) = \sum_{n=0}^{\infty} (n+k) a_n x^{n+k-1}, \quad y''(x) = \sum_{n=0}^{\infty} (n+k)(n+k-1) a_n x^{n+k-2}$$

Substituting for y, y' and y'' in the equation, we obtain

$$8\sum_{n=0}^{\infty}(n+k)(n+k-1)a_n x^{n+k} + 2\sum_{n=0}^{\infty}(n+k)a_n x^{n+k} + \sum_{n=0}^{\infty} a_n x^{n+k+1} + \sum_{n=0}^{\infty} a_n x^{n+k} = 0$$

Equating the coefficient of each power of x to zero, we obtain

$$[8k(k-1) + 2k + 1]\,a_0 = 0 \quad \ldots\ldots\ldots i$$

$$[8(n+k)(n+k-1) + 2(n+k) + 1]\,a_n + a_{n-1} = 0 \ldots . ii$$

$a_0 \neq 0$

The indicial equation $8k^2 - 6k + 1 = 0$ gives $k = \frac{1}{2}, \frac{1}{4}$

Substituting $k = \frac{1}{2}$ in ii, we obtain the recurrence relation

$$a_n = \frac{-1}{2n(4n+1)}\, a_{n-1}$$

$$a_1 = \frac{-1}{(2)(5)} a_0, \quad a_2 = \frac{-1}{(4)(9)} a_1 = \frac{1}{2.4.5.9} a_0 = \frac{1}{2^2(2!)\,5.9} a_0$$

3-33

Use the method of Frobenius to obtain two linearly independent series solutions about the regular singular point $x_0 = 0$ for

$$2xy'' - y' + 2y = 0$$

****

Since $x=0$ is a regular singular point, we assume that $y = \sum_{n=0}^{\infty} c_n x^{n+r}$. So $y' = \sum_{n=0}^{\infty} (n+r) c_n x^{n+r-1}$ and $y'' = \sum_{n=0}^{\infty} (n+r-1)(n+r) c_n x^{n+r-2}$

Substituting,

$$2\sum_{n=0}^{\infty} (n+r-1)(n+r) c_n x^{n+r-1} - \sum_{n=0}^{\infty} (n+r) c_n x^{n+r-1} + 2\sum_{n=0}^{\infty} c_n x^{n+r} = 0$$

(Let $k = n-1$) (Let $k = n-1$) (Let $k = n$)

Shifting indices, and simplifying we have

$$x^r \left\{ 2(r-1) r c_0 x^{-1} - r c_0 x^{-1} + \sum_{k=0}^{\infty} \left[2(k+r)(k+r+1) c_{k+1} - (k+r+1) c_{k+1} + 2c_k \right] x^k \right\} = 0.$$

If $c_0 \neq 0$, then $2r^2 - 3r = 0 \Rightarrow r = 0$ or $r = 3/2$

and $$c_{k+1} = \frac{-2c_k}{2(k+r)(k+r+1) - (k+r+1)}$$

For $r=0$ we have $y_1 = c_0 (1 + 2x - 2x^2 + \cdots)$

and for $r = 3/2$, $y_2 = c_0 x^{3/2} \left(1 - \frac{2}{5}x + \frac{2}{35}x^2 + \cdots\right)$

3-35

Find the general solution and the interval of convergence of the series solution. $2xy'' + (1+2x)y' - 5y = 0$

****

The only singular point is $x=0$. So the solution is valid for $x>0$.

Let $y = \sum_{n=0}^{\infty} a_n x^{n+c}$, $y' = \sum_{n=0}^{\infty} (n+c) a_n x^{n+c-1}$

$$y'' = \sum_{n=0}^{\infty} (n+c-1)(n+c) a_n x^{n+c-2}$$

Substitute these into the original equation:

$$\sum_{n=0}^{\infty} 2(n+c-1)(n+c) a_n x^{n+c-1} + \sum_{n=0}^{\infty} (n+c) a_n x^{n+c-1}$$

$$+ \sum_{n=0}^{\infty} 2(n+c) a_n x^{n+c} - \sum_{n=0}^{\infty} 5 a_n x^{n+c} = 0$$

$n/n-1$ $\quad$ $n/n-1$

$$\sum_{n=0}^{\infty} \left[2(n+c-1)(n+c) + (n+c)\right] a_n x^{n+c-1} = \sum_{n=1}^{\infty} \left[5 - 2(n+c-1)\right] a_{n-1} x^{n+c-1}$$

Let $n=0$. $\quad 2(c-1)c + c = 0$

$$c(2c-1) = 0 \qquad c = \frac{1}{2}, 0.$$

When $c = \frac{1}{2}$ and $n \geq 1$:

$$\left[2(n-\tfrac{1}{2})(n+\tfrac{1}{2}) + (n+\tfrac{1}{2})\right] a_n = \left[5 - 2(n-\tfrac{1}{2})\right] a_{n-1}$$

$$a_n = \frac{-2(n-3)\, a_{n-1}}{n(2n+1)}$$

$$a_3 = \frac{-1}{(6)(13)} a_2 = \frac{-1}{2.4.6.5.9.13} a_0 = \frac{-1}{2^3(3!)\,5.9.13} a_0$$

Therefore the Frobenius series corresponding to $K = 1/2$ is

$$Y_1(x) = a_0 x^{1/2} \sum_n \frac{(-1)^n}{2^n(n!)\,5.9.\ldots(4n+1)} x^n$$

Substituting $k = \frac{1}{4}$ in ii gives

$$a_n = \frac{-1}{2n(4n-1)} a_{n-1}$$

so $a_1 = \frac{-1}{2.3} a_0$, $a_2 = \frac{-1}{4.7} = \frac{1}{2.4.3.7} a_0$

$$= \frac{1}{2^2(2!)3.7} a_0$$

$$a_3 = \frac{-1}{6 \cdot 11} a_2 = \frac{-1}{2.4.6.3.7.11} a_0$$

$$= \frac{-1}{2^3(3!)3.7.11} a_0$$

$$a_n = \frac{(-1)^n}{2^n(n!)3.7.\ldots(4n-1)} a_0$$

Therefore the Frobenius series corresponding to $K = 1/4$ is

$$Y_2(x) = a_0 x^{1/4}\left[1 + \sum_{n=1}^{\infty} \frac{(-1)^n}{2^n(n!)3.7.\ldots(4n-1)} a_n\right]$$

The general solution is given by

$Y(x) = C_1 Y_1(x) + C_2 Y_2(x)$, where C_1, C_2 are arbitrary constants.

Note that in this example the difference between the roots of the indicial equation, $1/2 - 1/4 = 1/4$, is not an integer.

3-36

Solve: $2xy'' + (1 - 2x^2)y' - 4xy = 0.$

****

AN INFINITE SERIES SOLUTION IS REQUIRED. $X=0$ IS A REGULAR SINGULAR POINT.

$$\therefore y = \sum_{m=0}^{\infty} a_m x^{m+c}, \quad y' = \sum_{m=0}^{\infty} (m+c) a_m x^{m+c-1}, \quad y'' = \sum_{m=0}^{\infty} (m+c)(m+c-1) a_m x^{m+c-2}.$$

SUBSTITUTING INTO THE LEFT SIDE OF THE GIVEN DIFFERENTIAL EQUATION:

$$\sum_{m=0}^{\infty} 2(m+c)(m+c-1) a_m x^{m+c-1} + \sum_{m=0}^{\infty} (m+c) a_m x^{m+c-1} - \sum_{m=0}^{\infty} 2(m+c) a_m x^{m+c+1} - \sum_{m=0}^{\infty} 4 a_m x^{m+c+1}.$$

COMBINE LIKE EXPONENTS: $\sum_{m=0}^{\infty} (m+c)(2m+2c-1) a_m x^{m+c-1} - \sum_{m=0}^{\infty} (2m+2c+4) a_m x^{m+c+1}.$

REINDEX 2ND SUM: $\sum_{m=0}^{\infty} (m+c)(2m+2c-1) a_m x^{m+c-1} - \sum_{m=2}^{\infty} (2m+2c) a_{m-2} x^{m+c-1}$

WRITE 1ST TWO TERMS OF 1ST SUM SEPARATELY, THEN COMBINE SUMS:

$$c(2c-1) a_0 x^{c-1} + (c+1)(2c+1) a_1 x^{c} + \sum_{n=2}^{\infty} \left[(n+c)(2n+2c-1) a_n - (2n+2c) a_{n-2} \right] x^{n+c-1}$$

SETTING THE PREVIOUS LINE TO ZERO REQUIRES $c(2c-1) a_0 = 0$, $(c+1)(2c+1) a_1 = 0$, AND $(n+c)(2n+2c-1) a_n - (2n+2c) a_{n-2} = 0$, $n \geq 2$. WE CAN CHOOSE $c=0$ OR $c=\frac{1}{2}$, LEAVING a_0 ARBITRARY, AND FORCING $a_1 = 0$ OR CHOOSE $c=-1$ OR $c=-\frac{1}{2}$, LEAVING a_1 ARBITRARY AND FORCING $a_0 = 0$. EITHER CHOICE PRODUCES A SOLUTION FOR EACH c. EITHER WAY,

$$(n+c)(2n+2c-1) a_n - 2(n+c) a_{n-2} = 0, \qquad n \geq 2.$$

$$\therefore a_n = \frac{2}{2n+2c-1} a_{n-2}, \quad 2 \leq n. \quad (n+c) \text{ WILL NOT VANISH.}$$

$$a_2 = \frac{2}{2c+3} a_0. \quad a_4 = \frac{2}{2c+7} a_2 = \frac{2^2}{(2c+3)(2c+7)} a_0, \text{ ETC.}$$

GENERALIZING, $a_{2k} = \dfrac{2^k}{(2c+3)(2c+7)\cdots(2c+4k-1)}, \quad 1 \leq k.$

$$a_3 = \frac{2}{2c+5} a_1 \quad a_5 = \frac{2}{2c+9} a_3 = \frac{2^2}{(2c+5)(2c+9)} a_1, \text{ ETC.}$$

$$a_1 = \frac{-2(-2)a_0}{(1)(3)} = \frac{4}{3}a_0 \qquad a_2 = \frac{(-2)(-1)a_1}{(2)(5)} = \frac{1}{5}a_1 = \frac{4}{15}a_0$$

$a_n = 0$ if $n \geq 3$. So $y_1 = a_0\left[x^{\frac{1}{2}} + \frac{4}{3}x^{\frac{3}{2}} + \frac{4}{15}x^{\frac{5}{2}}\right]$

When $c=0$ and $n \geq 1$, $\left[2(n-1)n+n\right]a_n = \left[5-2(n-1)\right]a_{n-1}$

$$a_n = -\frac{(2n-7)a_{n-1}}{n(2n-1)}$$

$$a_1 = \frac{-(-5)a_0}{(1)(1)}$$

$$a_2 = -\frac{(-3)a_1}{(2)(3)}$$

$\vdots$

Multiplying:

$$a_n = \frac{(-1)^n\left[(-5)(-3)(-1)\cdots(2n-7)\right]a_0}{n!\left[1\cdot 3\cdot 5\cdots(2n-7)(2n-5)(2n-3)(2n-1)\right]}$$

$$= \frac{(-1)^{n+1}(15)a_0}{n!\,(2n-5)(2n-3)(2n-1)}$$

So $y_2 = a_0\left[1 + \sum_{n=1}^{\infty}\frac{(-1)^{n+1}15x^n}{n!\,(2n-5)(2n-3)(2n-1)}\right]$

The general solution is

$$y = c_1\left[x^{\frac{1}{2}} + \frac{4}{3}x^{\frac{3}{2}} + \frac{4}{15}x^{\frac{5}{2}}\right] + c_2\left[1 + \sum_{n=1}^{\infty}\frac{(-1)^{n+1}15x^n}{n!\,(2n-5)(2n-3)(2n-1)}\right]$$

3-37

Produce the Frobenius Series solution to the following second order ordinary differential equation about the 'regular singular point' $x_o = 0$:

$3x \cdot y'' + y' + 2y = 0$; $y = y(x)$.

****

Assuming $y = \sum_{n=0}^{\infty} c_n \cdot x^{r+n}$, the differential equation reads:

$$\left[\sum_{n=0}^{\infty}\left(3c_n\cdot(r+n)\cdot(r+n-1)\cdot x^{r+n-1}\right) + \sum_{0}^{\infty}\left(c_n\cdot(r+n)\cdot x^{r+n-1}\right) + \sum_{0}^{\infty}\left(2c_n\cdot x^{r+n}\right)\right]=0,$$

or

$$\left\{\left[3c_0 r(r-1) + c_0\cdot r\right]\cdot x^{r-1} + \sum_{1}^{\infty}\left(3c_n\cdot(r+n)\cdot(r+n-1)\cdot x^{r+n-1}\right)\right.$$
$$\left. + \sum_{1}^{\infty}\left(c_n\cdot(r+n)\cdot x^{r+n-1}\right) + \sum_{0}^{\infty}\left(2c_n\cdot x^{r+n}\right)\right\} = 0,$$

or

$$\left[3c_0 r\cdot(r-1) + c_0\cdot r\right]\cdot x^{r-1} + \sum_{n=0}^{\infty}\left\{\left[3\cdot c_{n+1}\cdot(r+n+1)\cdot(r+n) + c_{n+1}\cdot(r+n+1) + 2c_n\right]\cdot x^{r+n}\right\}$$
$$= 0.$$

Viewing this latter equation as $\sum_{m=m_o}^{\infty}\left(\alpha_m \cdot x^m\right) = 0$, the equation can be satisfied by setting each coefficient α_m equal to zero.

Equating the first coefficient equal to zero gives the indicial equation, $c_0 \cdot r\cdot(3r-2) = 0$. If $c_0 \neq 0$, then $\begin{cases} r_1 = 0 \\ r_2 = \frac{2}{3} \end{cases}$ Indicial Roots.

Equating the coefficient of the arbitrary power term equal to zero gives the recurrence relation, $c_{n+1} = \dfrac{-2\cdot c_n}{(r+n+1)\cdot[1+3(r+n)]}$;

$n \geq 0$.

With $r_1 = 0$, $\hat{c}_{n+1} = \dfrac{-2\cdot \hat{c}_n}{(n+1)\cdot(3n+1)}$.

GENERALIZING, $a_{2K+1} = \dfrac{2^K}{(2C+5)(2C+9)\cdots(2C+4K+1)}\,a_1, \quad 1 \le K.$

NOW THE CHOICE MUST BE MADE. WE ARBITRARILY CHOOSE $a_0 = 0$, THUS FORCING $C = -1$ OR $C = -\frac{1}{2}$. THEN $a_{2K} = 0$.

WITH $C = -1$, $a_{2K+1} = \dfrac{2^K}{3 \cdot 7 \cdot \cdots (4K-1)}\,a_1, \quad 1 \le K.$

WITH $C = -\frac{1}{2}$, $a_{2K+1} = \dfrac{2^K}{4 \cdot 8 \cdot 12 \cdot \cdots (4K)}\,a_1, \quad 1 \le K$

$$= \frac{2^K}{(2^2 \cdot 1)(2^2 \cdot 2)(2^2 \cdot 3)\cdots(2^2 \cdot K)}\,a_1$$

$$= \frac{2^K}{2^{2K} \cdot K!}\,a_1 = \frac{1}{2^K \cdot K!}\,a_1.$$

$$\sum_{n=0}^{\infty} a_n X^{n+C} = a_0 X^C + a_1 X^{C+1} + \sum_{K=1}^{\infty} a_{2K} X^{2K+C} + \sum_{K=1}^{\infty} a_{2K+1} X^{2K+1+C}$$

IS AN IDENTITY.

IN THIS PROBLEM, $C = -1 \Rightarrow y_1 = a_1 + \displaystyle\sum_{K=1}^{\infty} \frac{2^K a_1}{3 \cdot 7 \cdot \cdots (4K-1)} X^{2K}.$

$$C = -\tfrac{1}{2} \Rightarrow y_2 = a_1 X^{1/2} + \sum_{K=1}^{\infty} \frac{a_1}{2^K \cdot K!} X^{2K+1/2}.$$

y_1 AND y_2 ARE LINEARLY INDEPENDENT, SO THE GENERAL SOLUTION CAN BE WRITTEN AS $y = K_1 y_1 + K_2 y_2$ WHERE K_1 AND K_2 ARE ARBITRARY.

3-38

Determine the general solution of $2xy'' - y' + y = 0$ in terms of Frobenius series about the origin. Display the first four terms of any series.

****

Assume Frobenius series solution(s) $y = \sum a_k x^{k+c}$,

where (for convenience) $a_k = 0$ for all k negative.

Then $y' = \sum (k+c) a_k x^{k+c-1}$ and

$y'' = \sum (k+c)(k+c-1) a_k x^{k+c-2}$. Hence

$2xy'' = \sum 2(k+c)(k+c-1) a_k x^{k+c-1}$. Since both the $2xy''$ and y' terms have the same exponent $k+c-1$, it is necessary to re-write the y term: $y = \sum a_k x^{k+c} = \sum a_{k-1} x^{k+c-1}$ (letting $k = k-1$ in this term only).

Now we can substitute the three series into the equation and factor out x^{k+c-1}:

$$\sum \left[2(k+c)(k+c-1) a_k - (k+c) a_k + a_{k-1} \right] x^{k+c-1} = 0.$$

The bracketed coefficients must be zero for all k, and simplifying, we obtain

$$a_k (k+c)(2k+2c-3) + a_{k-1} = 0.$$

Letting $k = 0$ makes $a_{k-1} = a_{-1} = 0$,

Then $\hat{r}_1 = \dfrac{-2\hat{r}_0}{1!\cdot 1}$, $\hat{r}_2 = \dfrac{-2\hat{r}_1}{2\cdot 4} = \dfrac{(-2)^2\cdot\hat{r}_0}{2!\cdot 1\cdot 4}$, $\hat{r}_3 = \dfrac{-2\hat{r}_2}{3\cdot 7} = \dfrac{(-2)^3\cdot\hat{r}_0}{3!\cdot 1\cdot 4\cdot 7}$,

...

Thus $y_1 = x^{r_1}\cdot(\hat{r}_0 + \hat{r}_1\cdot x + \hat{r}_2\cdot x^2 + \cdots)$

$= \hat{r}_0\cdot\left[1 - \dfrac{2}{1!\cdot 1}x + \dfrac{2^2}{2!\cdot 1\cdot 4}\cdot x^2 - \dfrac{2^3}{3!\cdot 1\cdot 4\cdot 7}\cdot x^3 + -\cdots\right]$, or

$$y_1 = \hat{r}_0\cdot\left[1 + \sum_{n=1}^{\infty}\left\{\frac{(-2)^n\cdot x^n}{n!\cdot(1\cdot 4\cdot 7\cdots(3n-2))}\right\}\right].$$

With $r_2 = \frac{2}{3}$, $\tilde{r}_{n+1} = \dfrac{-2\cdot\tilde{r}_n}{(n+1)(3n+5)}$.

Here $\tilde{r}_1 = \dfrac{-2\tilde{r}_0}{1!\cdot 5}$, $\tilde{r}_2 = \dfrac{(-2)^2\cdot\tilde{r}_0}{2!\cdot 5\cdot 8}$, $\tilde{r}_3 = \dfrac{(-2)^3\cdot\tilde{r}_0}{3!\cdot 5\cdot 8\cdot 11}$, $\cdots$, giving

$y_2 = x^{r_2}\cdot(\tilde{r}_0 + \tilde{r}_1\cdot x + \tilde{r}_2\cdot x^2 + \cdots) = \tilde{r}_0\cdot x^{\frac{2}{3}}\cdot\left[1 - \dfrac{2}{1!\cdot 5}x + \dfrac{2^2}{2!\cdot 5\cdot 8}\cdot x^2 + \cdots\right]$, or

$$y_2 = \tilde{r}_0\cdot x^{\frac{2}{3}}\cdot\left[1 + \sum_{n=1}^{\infty}\left\{\frac{(-2)^n\cdot x^n}{n!\cdot 5\cdot 8\cdot 11\cdots(3n+2)}\right\}\right].$$

Then, the complete or general solution (Frobenius type) to the given differential equation is $y = y_1 + y_2$.

To reveal the constructive freedom in the analysis, suppose $r_0 \equiv 0$. Then, setting the coefficient of the first-assumed non-zero term to zero gives the indicial equation, $r_1\cdot(r+1)\cdot(1+3r) = 0$.

Assume $r_1 \neq 0$ to produce indicial roots $r_1 = -1$, $r_2 = -\frac{1}{3}$.

Producing the associated series for these roots will yield the same general solution as described above.

3-39

Solve $x^2(x+2)\,y'' - xy' + (x+1)y = 0$ on $x > 0$.

****

$$y'' - \frac{1}{x(x+2)}\,y' + \frac{1+x}{x^2(x+2)}\,y = 0$$

$$\lim_{x\to 0} x\cdot\frac{-1}{x(x+2)} = \lim_{x\to 0}\left(-\frac{1}{x+2}\right) = -\frac{1}{2}$$

$$\lim_{x\to 0} x^2\cdot\frac{1+x}{x^2(x+2)} = \lim_{x\to 0}\frac{1+x}{x+2} = \frac{1}{2}$$

Hence $x=0$ is a regular singular point and we can look for Frobenius Series solutions.

$$y = x^r\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^{n+r}, \quad a_0 \neq 0.$$

$$y' = \sum_{n=0}^{\infty}(n+r)\,a_n x^{n+r-1}, \quad y'' = \sum_{n=0}^{\infty}(n+r)(n+r-1)\,a_n x^{n+r-2}$$

Putting these into the original equation gives

$$x^2(x+2)\sum_{n=0}^{\infty}(n+r)(n+r-1)x^{n+r-2} - x\sum_{n=0}^{\infty}(n+r)a_n x^{n+r-1}$$

$$+(x+1)\sum_{n=0}^{\infty}a_n x^{n+r} = 0.$$

$$(x^3+2x^2)\sum_{n=0}^{\infty}(n+r)(n+r-1)x^{n+r-2} - x\sum_{n=0}^{\infty}(n+r)a_n x^{n+r-1}$$

$$+(x+1)\sum_{n=0}^{\infty}a_n x^{n+r} = 0.$$

so we get the "indicial equation" $a_0 c(2c-3) = 0$. This has two roots, $c = 0$ and $c = 3/2$.

In the case $c = 0$ we will have just a power series, with recurrence formula $a_k = -\frac{a_{k-1}}{k(2k-3)}$. This gives $a_1 = a_0$, $a_2 = -a_0/2$, and $a_3 = a_0/18$. Hence one solution is $y_1 = a_0(1 + x - \frac{1}{2}x^2 + \frac{1}{18}x^3 - \dots)$.

In the case $c = 3/2$ we have the recurrence formula $a_k = -\frac{a_{k-1}}{k(2k+3)}$ This gives $a_1 = -a_0/5$, $a_2 = a_0/70$ and $a_3 = -a_0/1890$. Hence the other independent solution is the Frobenius-type series $y_2 = a_0 x^{3/2}(1 - \frac{1}{5}x + \frac{1}{70}x^2 - \frac{1}{1890}x^3 + \dots)$.

Thus, the general solution is the linear combination $y = c_1 y_1 + c_2 y_2$.

$$(2r^2-3r+1)\,a_0 = 0$$

$$\left[(k+r-1)(k+r-2)+1\right]a_{k-1} + \left[2(k+r)(k+r-1)-(k+r)+1\right]a_k = 0$$
$$k = 1, 2, 3, \cdots$$

Since $a_0 \neq 0$ we have the indicial eq for r

$$2r^2-3r+1 = 0 \qquad (2r-1)(r-1) = 0 \qquad r = 1, \tfrac{1}{2}$$

these do not differ by an integer, hence the general solution is of the form

$$y = C_1 x \sum_{n=0}^{\infty} a_n x^n + C_2 x^{\frac{1}{2}} \sum_{n=0}^{\infty} a_n^* x^n.$$

$$a_k = -\frac{(k+r-1)(k+r-2)+1}{2(k+r)(k+r-1)-(k+r)+1}\,a_{k-1}$$
$$k = 1, 2, 3, \cdots$$

gives two recurrence relations; one for a_n $(r=1)$ and one for a_n^* $(r=\tfrac{1}{2})$.

$\underline{r=1}$

$$a_k = -\frac{k(k-1)+1}{2k(k+1)-(k+1)+1}\,a_{k-1}$$

$$a_k = -\frac{k(k-1)+1}{k(2k+1)}\,a_{k-1}$$

a_0 and a_0^* are arbitrary (but not zero).

$\underline{r=1}$

a_0 = arbitrary

$a_1 = -\frac{1}{3}a_0$

$a_2 = -\frac{3}{10}a_1 = -\frac{3}{10}\left(-\frac{1}{3}a_0\right) = \frac{1}{10}a_0$

$\underline{r=\frac{1}{2}}$

$$a_k^* = -\frac{(k-\frac{1}{2})(k-\frac{3}{2})+1}{2(k+\frac{1}{2})(k-\frac{1}{2})-(k+\frac{1}{2})+1}\,a_{k-1}^*$$

$$a_k^* = -\frac{4(2k-1)(2k-3)+1}{8(2k+1)(2k-1)-2(2k+1)+1}\,a_{k-1}^*$$

$\underline{r=\frac{1}{2}}$

a_0^* = arbitrary

$a_1^* = -\frac{-3}{19}a_0^* = \frac{3}{19}a_0^*$

$a_2^* = -\frac{13}{111}a_1^* = -\frac{13(3)}{111(19)}a_0^*$

$$\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r+1} + \sum_{n=0}^{\infty}2(n+r)(n+r-1)a_n x^{n+r}$$
$$-\sum_{n=0}^{\infty}(n+r)a_n x^{n+r} + \sum_{n=0}^{\infty}a_n x^{n+r+1} + \sum_{n=0}^{\infty}a_n x^{n+r} = 0.$$

Let $n=k-1$ in the 1st sum, $n=k$ in the 2nd, $n=k$ in the 3rd, $n=k-1$ in the 4th and $n=k$ in the 5th.

$$\sum_{k=1}^{\infty}(k+r-1)(k+r-2)a_{k-1}x^{k+r} + \sum_{k=0}^{\infty}2(k+r)(k+r-1)a_k x^{k+r}$$
$$-\sum_{k=0}^{\infty}(k+r)a_k x^{k+r} + \sum_{k=1}^{\infty}a_{k-1}x^{k+r} + \sum_{k=0}^{\infty}a_k x^{k+r} = 0.$$

$$\sum_{k=1}^{\infty}(k+r-1)(k+r-2)a_{k-1}x^{k+r} + 2r(r-1)a_0 x^r$$
$$+\sum_{k=1}^{\infty}2(k+r)(k+r-1)a_k x^{k+r} - ra_0 x^r - \sum_{k=1}^{\infty}(k+r)a_k x^{k+r}$$
$$+\sum_{k=1}^{\infty}a_{k-1}x^{k+r} + a_0 x^r + \sum_{k=1}^{\infty}a_k x^{k+r} = 0.$$

$$[2r(r-1) - r + 1]a_0 x^r$$
$$+\sum_{k=1}^{\infty}\Big[(k+r-1)(k+r-2)a_{k-1} + 2(k+r)(k+r-1)a_k - (k+r)a_k$$
$$+ a_{k-1} + a_k\Big]x^{k+r} = 0.$$

$$(2r^2-3r+1)a_0 x^r + \sum_{k=1}^{\infty}\Big\{\big[(k+r-1)(k+r-2)+1\big]a_{k-1}$$
$$+\big[2(k+r)(k+r-1) - (k+r)+1\big]a_k\Big\}x^{k+r} = 0.$$

Equate coefficients of like powers of x on the left and right (remember: $0 = \sum_{k\geq 0}^{\infty} 0\cdot x^{k+r} = 0\cdot x^r + \sum_{k=1}^{\infty} 0\cdot x^{k+r}$).

3-40

Show that for the following equation

$$2x^2 y''(x) + x(2x + 1) y'(x) - y(x) = 0,$$

the point $x = 0$ is a regular singular point and then find the solution in a power series by Frobenius method.

****

Rewrite the equation as

$$y''(x) + \frac{\frac{1}{2}+x}{x} y'(x) - \frac{\frac{1}{2}}{x^2} y(x) = 0$$

and notice that the coefficients of $y'(x)$ and $y(x)$, call them $P(x)$ and $Q(x)$ respectively, are singular at $x=0$. However $xP(x) = \frac{1}{2} + x$ and $x^2Q(x) = -\frac{1}{2}$ are both regular at $x=0$. Hence $x=0$ is a regular singular point. We now assume the Frobenius series solution of the form

$$y(x) = x^m (a_0 + a_1 x + a_2 x^2 + \cdots), \quad a_0 \neq 0.$$

Their derivatives are

$$y'(x) = a_0 m x^{m-1} + a_1 (m+1) x^m + a_2 (m+2) x^{m+1} + \cdots$$

and

$$y''(x) = a_0 m(m-1) x^{m-2} + a_1 (m+1) m x^{m-1} + a_2 (m+2)(m+1) x^m + \cdots$$

Substitute the 3 series into the differential equation and cancell their common factor x^{m-2}. The result becomes

$$a_0 m(m-1) + a_1 (m+1) m x + a_2 (m+2)(m+1) x^2 + \cdots$$
$$+ (\tfrac{1}{2} + x)\left[a_0 m + a_1 (m+1) x + a_2 (m+2) x^2 + \cdots\right]$$
$$- \tfrac{1}{2}\left[a_0 + a_1 x + a_2 x^2 + \cdots\right] = 0$$

$$a_3 = -\frac{7}{21} a_2 = -\frac{1}{3}\left(\frac{1}{10} a_0\right)$$

$$= -\frac{1}{3 \cdot 10} a_0$$

etc.

$$= -\frac{3 \cdot 13}{19 \cdot 111} a_0^*$$

$$a_3^* = -\frac{61}{267} a_2^* = -\frac{61}{267}\left(-\frac{3 \cdot 13}{19 \cdot 111} a_0^*\right)$$

$$= \frac{3 \cdot 13 \cdot 61}{19 \cdot 111 \cdot 267} a_0^*$$

etc.

$$y_1 = x\left(a_0 - \frac{1}{3} a_0 x + \frac{1}{10} a_0 x^2 - \frac{1}{3 \cdot 10} a_0 x^3 + \cdots\right)$$

$$y_2 = x^{\frac{1}{2}}\left(a_0^* + \frac{13}{19} a_0^* x - \frac{3 \cdot 13}{19 \cdot 111} a_0^* x^2 + \frac{3 \cdot 13 \cdot 61}{19 \cdot 111 \cdot 267} a_0^* x^3 + \cdots\right)$$

$$y_1 = a_0 x\left(1 - \frac{1}{3} x + \frac{1}{10} x^2 - \frac{1}{3 \cdot 10} x^3 + \cdots\right)$$

$$y_2 = a_0^* x^{\frac{1}{2}}\left(1 + \frac{3}{19} x - \frac{3 \cdot 13}{19 \cdot 111} x^2 + \frac{3 \cdot 13 \cdot 61}{19 \cdot 111 \cdot 267} x^3 + \cdots\right)$$

The general solution can be written as $y = y_1 + y_2$ where a_0 and a_0^* are thought of as the arbitrary constants c_1 and c_2, respectively.

$$y = c_1 x\left(1 - \frac{1}{3} x + \frac{1}{10} x^2 - \frac{1}{3 \cdot 10} x^3 + \cdots\right)$$
$$+ c_2 x^{\frac{1}{2}}\left(1 + \frac{3}{19} x - \frac{3 \cdot 13}{19 \cdot 111} x^2 + \frac{3 \cdot 13 \cdot 61}{19 \cdot 111 \cdot 267} x^3 + \cdots\right).$$

3-41

The differential equation $2xy'' + y' - xy = 0$ has a Frobenius type series solution of the form $\sum a_k x^{k+c}$ about $x = 0$. Determine the roots of the indicial equation.

****

Assuming the necessary properties for the series we have

$$y = \sum_{k=0}^{\infty} a_k x^{k+c}$$

$$y' = \sum_{k=0}^{\infty} (k+c) a_k x^{k+c-1}$$

$$y'' = \sum_{k=0}^{\infty} (k+c-1)(k+c) a_k x^{k+c-2}$$

substitution in the differential equation yields

$$2x \sum_{k=0}^{\infty} (k+c-1)(k+c) a_k + \sum_{k=0}^{\infty} (k+c) a_k x^{k+c-1} - x \sum_{k=0}^{\infty} a_k x^{k+c} = 0$$

$$\sum_{k=0}^{\infty} 2(k+c-1)(k+c) a_k x^{k+c-1} + \sum_{k=0}^{\infty} (k+c) a_k x^{k+c-1} - \sum_{k=0}^{\infty} a_k x^{k+c-1} = 0$$

shift indicies by setting $k = k+1$ in the first and second terms and $k = k-1$ in the third term. note: k is a dummy variable

we obtain

$$\sum_{k=0}^{\infty} 2(k+c)(k+c+1) a_{k+1} x^{k+c} + \sum_{k=0}^{\infty} (k+1+c) a_{k+1} x^{k+c} - \sum_{k=1}^{\infty} a_{k-1} x^{k+c} = 0$$

or $$\sum_{k=0}^{\infty} \left[2(k+c)(k+c+1) a_{k+1} + (k+c+1) a_{k+1} - a_{k-1}\right] x^{k+c} = 0$$

so that $$\left[2(k+c)(k+c+1) + (k+c+1) a_{k+1} - a_{k-1}\right] = 0$$

$$(2(k+c)+1)(k+c+1) a_{k+1} - a_{k-1} = 0$$

Set $k = -1$ and obtain $(2c-1)(c) a_0 = 0 \quad a_0 \neq 0$

so $\underline{\underline{c = 1/2}}$ and $\underline{\underline{c = 0}}$ (note $a_{-2} = 0$).

Combine like power terms of x and equate the coefficient of each power of x to zero.

$$a_0\left[m(m-1)+\tfrac{1}{2}m-\tfrac{1}{2}\right]=0$$

$$a_1\left[(m+1)m+\tfrac{1}{2}(m+1)-\tfrac{1}{2}\right]+a_0 m=0$$

$$a_2\left[(m+2)(m+1)+\tfrac{1}{2}(m+2)-\tfrac{1}{2}\right]+a_1(m+1)=0 \quad \text{etc.}$$

Since we assumed $a_0 \neq 0$, we obtain the indicial equation

$$m(m-1)+\tfrac{1}{2}m-\tfrac{1}{2}=0$$

giving the two roots

$$m_1 = 1 \text{ and } m_2 = -\tfrac{1}{2}.$$

For $m_1 = 1$, $a_1 = \dfrac{a_0}{2\cdot 1+\frac{1}{2}\cdot 2-\frac{1}{2}} = \dfrac{-2}{5}a_0$, $a_2 = \dfrac{4}{35}a_0$ etc

and for $m_2 = -\frac{1}{2}$, $a_1 = \dfrac{\frac{1}{2}a_0}{\frac{1}{2}\left(-\frac{1}{2}\right)+\frac{1}{2}\cdot\frac{1}{2}-\frac{1}{2}} = -a_0$, $a_2 = \frac{1}{2}a_0$.

Without loss of generality, we take $a_0 = 1$

$$y_1(x) = x\left(1-\tfrac{2}{5}x+\tfrac{4}{35}x^2+\cdots\right)$$

$$y_2(x) = x^{-1/2}\left(1-x+\tfrac{1}{2}x^2+\cdots\right)$$

The two solutions are independent for $x>0$. The general solution is then

$$y(x) = C_1 x\left(1-\tfrac{2}{5}x+\tfrac{4}{35}x^2+\cdots\right) + C_2 x^{-1/2}\left(1-x+\tfrac{1}{2}x^2+\cdots\right).$$

In general, $a_n = -\dfrac{(n+c-2)\,a_{n-1}}{(n+c-1)^2}$ when $n \geq 1$. Let $a_0 = 1$

$$a_1 = -\frac{(c-1)}{c^2}a_0 \qquad a_2 = -\frac{c\,a_1}{(c+1)^2} \quad \cdots$$

Multiplying, $a_n = \dfrac{(-1)^n\left[(c-1)c(c+1)\cdots(c+n-2)\right]}{\left[c(c+1)\cdots(c+n-1)\right]^2}$

So $a_n = \dfrac{(-1)^n(c-1)}{(c+n-1)^2 c(c+1)\cdots(c+n-2)}$

$$a_n' = a_n\left[\frac{1}{c-1} - \frac{2}{c+n-1} - \frac{1}{c} - \frac{1}{c+1} - \cdots - \frac{1}{c+n-2}\right]$$

$$= \frac{(-1)^n}{(c+n-1)^2 c(c+1)\cdots(c+n-2)} + \frac{(-1)^n(c-1)}{(c+n-1)^2 c(c+1)\cdots(c+n-2)}\cdot$$

$$\left[\frac{-2}{c+n-1} - \frac{1}{c} - \cdots - \frac{1}{c+n-2}\right]$$

When $c = 1$, $a_n' = \dfrac{(-1)^n}{n \cdot n!}$

The general solution is

$$y = a_0 x + b_0\left(x \ln x + \sum_{n=1}^{\infty} \frac{(-1)^n x^{n+1}}{n \cdot n!}\right)$$

SECOND ORDER WITH $X = X_0$ A REGULAR SINGULAR POINT

EQUAL ROOTS

3-42

Find the general solution and the interval of convergence of the series solution. $x^2y'' + x(x-1)y' + (1-x)y = 0$

****

$x=0$ is the only singular point. The solution is valid for $x>0$.

Let $y = \sum_{n=0}^{\infty} a_n x^{n+c}$. Substitute into the differential equation.

$$\sum_{n=0}^{\infty}(n+c)(n+c-1)a_n x^{n+c} + \sum_{n=0}^{\infty}(n+c)a_n x^{n+c+1} - \sum_{n=0}^{\infty}(n+c)a_n x^{n+c}$$
$$+ \sum_{n=0}^{\infty} a_n x^{n+c} - \sum_{n=0}^{\infty} a_n x^{n+c+1} = 0$$

$$\sum_{n=0}^{\infty}\left[(n+c)(n+c-1) - (n+c) + 1\right]a_n x^{n+c} = -\sum_{n=1}^{\infty}\left[(n+c-1)\right]a_{n-1}x^{n+c}$$
$$\uparrow$$
$$(n+c-2)$$

Let $n=0$: $(c-1)^2 = 0 \quad c=1,1$

Let $c=1$ and $n\geq 1$: $n^2 a_n = -(n-1)a_{n-1} \quad a_n = -\dfrac{(n-1)a_{n-1}}{n^2}$

$a_n = 0$ if $n \geq 1$.

$\therefore y_1 = a_0 x^c = a_0 x$

by $y_2(x) = \frac{\partial}{\partial k} y(x,k)\Big|_{k=2}$ $y(x,k)$ is the Frobenius series $y(x)$ in which the coefficients a_n are functions of k (we do not substitute $k=2$ in a_n, not yet!)

Then $a_n = \frac{-4}{(n+k-2)^2} a_{n-1}$

$a_1 = \frac{-4}{(k-1)^2} a_0$, $a_2 = \frac{-4}{k^2} a_1 = \frac{4^2}{(k-1)^2 k^2} a_0$, $a_3 = -\frac{4^3}{(k-1)^2 k^2 (k+1)^2} a_0$

$a_n = \frac{(-1)^n 4^n}{(k-1)^2 k^2 (k+1)^2 \cdots (k+n-2)^2} a_0$

Therefore $y(x,k) = a_0\left[x^k - \frac{4}{(k-1)^2} x^{k+1} + \frac{4^2}{(k-1)^2 k^2} x^{k+2} - \frac{4^3}{(k-1)^2 k^2 (k+1)^2} x^{k+3} + \cdots\right]$

$\frac{\partial y(x,k)}{\partial k} = a_0\left[x^k \ln x - \frac{4}{(k-1)^2} x^{k+1} \ln x + \frac{4^2}{(k-1)^2 k^2} x^{k+2} \ln x - \frac{4^3}{(k-1)^2 k^2 (k+1)^2} x^{k+3} \ln x\right.$

$+ \cdots\cdots + \frac{8}{(k-1)^3} x^{k+1} + \frac{4^2(-2)}{(k-1)^3 k^2} x^{k+2} + \frac{4^2(-2)}{(k-1)^2 k^3} x^{k+2}$

$\left. - \frac{4^3(-2)}{(k-1)^3 k^2 (k+1)^2} x^{k+3} - \frac{4^3(-2)}{(k-1)^2 k^3 (k+1)^2} x^{k+3} - \frac{4^3(-2)}{(k-1)^2 k^2 (k+1)^3} x^{k+3} + \cdots\cdots\right]$

$y_2(x) = \frac{\partial y(x,k)}{\partial k}\Big|_{k=2} = a_0\left[x^2 \ln x - \frac{4}{1^2} x^3 \ln x + \frac{4^2}{1^2 \cdot 2^2} x^4 \ln x - \frac{4^3}{1^2 2^2 3^2} x^5 \ln x + \cdots\cdots\right.$

$+ \frac{8}{1^3} x^3 - \frac{4^2 \cdot 2}{1^3 \cdot 2^2} x^4 - \frac{4^2 \cdot 2}{1^2 \cdot 2^3} x^4 + \frac{4^3 \cdot 2}{1^3 2^2 3^2} x^5 + \frac{4^3 \cdot 2}{1^2 2^3 3^2} x^5$

$\left. + \frac{4^3 \cdot 2}{1^2 \cdot 2^2 \cdot 3^3} x^5 - \cdots\cdots\right]$

$y_2(x) = a_0 \ln x\left[x^2 - \frac{4}{1^2} x^3 + \frac{4^2}{1^2 \cdot 2^2} x^4 - \frac{4^3}{1^2 \cdot 2^2 \cdot 3^2} x^5 + \cdots\right]$

$+ a_0\left[8x^3 - 12x^4 + \frac{176}{27} x^5 - \cdots\right]$

Thus $y_2(x) = y_1(x) \ln x + a_0\left[8x^3 - 12x^4 + \frac{176}{27} x^5 - \cdots\right]$

The general solution of the given differential equation is $y(x) = c_1 y_1(x) + c_2 y(x)$, where c_1, c_2 are arbitrary constants.

3-43

Find the general solution of the differential equation

$$x^2y'' - 3xy' + 4(x+1)y = 0 \quad \text{near } x = 0.$$

****

$$P(x) = \frac{-3x}{x^2} = -\frac{3}{x}, \quad Q(x) = \frac{4(x+1)}{x^2} = \frac{4}{x} + \frac{4}{x^2}$$

$x=0$ is a singular point of the equation.
$(x-0)P(x) = -3$, $(x-0)^2Q(x) = 4(x+1)$ are analytic functions.
Therefore $x=0$ is a regular singularity of the equation.

Assume $y(x) = x^k \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^{n+k}$

$$y'(x) = \sum_{n=0}^{\infty} (n+k)a_n x^{n+k-1}, \quad y''(x) = \sum_{n=0}^{\infty} (n+k)(n+k-1)a_n x^{n+k-2}$$

Substituting for y, y' and y'' in the equation we obtain

$$x^2\sum_{n=0}^{\infty}(n+k)(n+k-1)a_n x^{n+k-2} - 3x\sum_{n=0}^{\infty}(n+k)a_n x^{n+k-1} + 4x\sum_{n=0}^{\infty}a_n x^{n+k} + 4\sum_{n=0}^{\infty}a_n x^{n+k} = 0$$

Equating the coefficient of each power of x to zero, we obtain

$$[k(k-1)-3k+4]a_0 = 0 \quad \ldots\ldots i$$

$$[(n+k)(n+k-1) - 3(n+k) + 4]a_n + 4a_{n-1} = 0 \quad \ldots\ldots ii$$

$a_0 \neq 0$

The indicial equation $k^2 - 4k + 4 = 0$ gives equal roots $k=2$

Substituting $k=2$ in ii we obtain the recurrence relation

$$a_n = -\frac{4}{n^2}a_{n-1}$$

$$a_1 = -\frac{4}{1^2}a_0, \quad a_2 = -\frac{4}{2^2}a_1 = \frac{4^2}{(2!)^2}a_0, \quad a_3 = -\frac{4}{3^2}a_2 = -\frac{4^3}{(3!)^2}$$

$$a_n = \frac{(-1)^n 4^n}{(n!)^2}$$

Therefore the Frobenius series corresponding to $k=2$ is

$$y_1(x) = a_0 x^2 \sum_{n=0}^{\infty} \frac{(-1)^n 4^n}{(n!)^2} x^n = a_0\left[x^2 - 4x^3 + 4x^4 - \frac{16}{9}x^5 + \ldots\ldots\right]$$

In this example, the indicial equation has equal roots = 2, therefore the other linearly independent solution is given

$$\text{So,} \quad \int \frac{e^{-\int P(x)\,dx}}{(y_1)^2}\,dx = \int \frac{x^{-1}}{1-\frac{1}{2}x^2+\frac{3}{32}x^4+\cdots}\,dx$$

$$= \int x^{-1}+\frac{1}{2}x+\frac{5}{32}x^3+\cdots\,dx$$

$$= \ln x + \frac{1}{4}x^2 + \frac{5}{128}x^4 + \cdots$$

$$\text{Thus,} \quad y_2 = y_1 \ln x + y_1\left[\frac{1}{4}x^2+\frac{5}{128}x^4+\cdots\right]$$

SECOND ORDER WITH X = X_0
A REGULAR SINGULAR POINT

DISTINCT ROOTS DIFFERING BY AN INTEGER

3-45

Find one solution of the equation

$$x^2y'' - xy' + (3/4 + x^2)y = 0 \qquad (1)$$

by using power series expansion near $x = 0$.

****

Write equation (1) in the form

$$y'' - \frac{1}{x}y' + \frac{3/4 + x^2}{x^2}y = 0 \qquad (2)$$

We see that $x=0$ is a regular singular point. The corresponding indicial equation

$$m(m-1) + p_0 m + q_0 = 0$$

3-44

Use the method of Frobenius to obtain two linearly independent series solutions about the regular singular point $x_0 = 0$ for

$$x^2y'' + xy' + x^2y = 0$$

****

Since $x = 0$ is a regular singular point, we assume that $y = \sum_{n=0}^{\infty} c_n x^{n+r}$. So $y' = \sum_{n=0}^{\infty} (n+r)\, c_n x^{n+r-1}$ and $y'' = \sum_{n=0}^{\infty} (n+r-1)(n+r) c_n x^{n+r-2}$

Substituting,

$$\sum_{n=0}^{\infty} (n+r-1)(n+r) c_n x^{n+r} + \sum_{n=0}^{\infty} (n+r) c_n x^{n+r} + \sum_{n=0}^{\infty} c_n x^{n+r+2} = 0.$$

(Let $k = n$) (Let $k = n$) (Let $k = n+2$)

Shifting indices and simplifying we have

$$x^r \left\{ [r(r-1)+r] c_0 + [r(r+1) + (r+1)] c_1 x + \sum_{k=2}^{\infty} [(k+r-1)(k+r) c_k + (k+r) c_k + c_{k-2}] x^k \right\} = 0.$$

If $c_0 \neq 0$, then $r^2 = 0 \Rightarrow r_1 = r_2 = 0$

and $c_k = -c_{k-2} / (k+r)^2$

For $r_1 = 0$ we have $y_1 = c_0 \left(1 - \frac{1}{4}x^2 + \frac{1}{64}x^4 - \cdots\right)$

Since $r_1 = r_2$ we obtain the second solution by applying the reduction of order formula

$$y_2 = y_1 \int \frac{e^{-\int P(x)\,dx}}{(y_1)^2}\, dx$$

Now, $e^{-\int P(x)\,dx} = e^{-\ln x} = x^{-1}$

and $y_1^2 = 1 - \frac{1}{2}x^2 + \frac{3}{32}x^4 + \cdots$

MISCELLANEOUS TECHNIQUES

3-46

It is well known that one solution of the equation

$$x(1-x)y'' + [c - (a+b+1)\cdot x]y' - aby = 0 \qquad (1)$$

near $x=0$ is the hypergeometric series $F(a, b, c, x)$. Use this result to find a solution (in terms of $F(a, b, c, x)$) for each of the following equations

a) $x(1-x)y'' + (3/2 - 2x)y' + 2y = 0$ near $x = 0$. (2)

b) $(x^2-1)y'' + (5x + 4)y' + 4y = 0$ near $x = -1$. (3)

****

a) Comparing (2) with (1), we find $c = \frac{3}{2}$, and $a+b+1 = 2$ $ab = -2$. Hence $a = -1$, $b = 2$. One solution of (2) can be written as $F(-1, 2, \frac{3}{2}, x)$.

b) Make change of variables to turn equation (3) into the form of (1). The coefficient of y'' in (3) (x^2-1) has two roots $x = -1$ and $x = 1$. So we let

$$t = \frac{x-(-1)}{1-(-1)} = \frac{x+1}{2}$$

Then $x = 2t-1$, $y' = \frac{dy}{dx} = \frac{dy}{dt}\cdot\frac{dt}{dx} = \frac{1}{2}\frac{dy}{dt}$, $y'' = \frac{1}{4}\frac{d^2y}{dt}$

and (3) becomes $t(1-t)\frac{d^2y}{dt^2} + (\frac{1}{2} - 5t)\frac{dy}{dt} - 4y = 0$. (4)

This equation is of the form of (1) with x replaced by t. Comparing (4) with (1), we have $c = \frac{1}{2}$, $a+b+1=5$ $ab = 4$. Hence $a=2$, $b=2$. Now (4) has a solution near $t = 0$: $F(2, 2, \frac{1}{2}, t)$

Therefore (3) has a solution near $x = -1$:

$$F(2, 2, \frac{1}{2}, \frac{x+1}{2}).$$

becomes $m^2 - 2m + \frac{3}{4} = 0$

with two different roots $\frac{1}{2}$ and $\frac{3}{2}$. Hence one of the solutions of (2) takes the form

$$y = x^{\frac{3}{2}} \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^{n+\frac{3}{2}}.$$

Then $$y' = \sum_{n=0}^{\infty} a_n (n+\tfrac{3}{2}) x^{n+\frac{1}{2}}$$

$$y'' = \sum_{n=0}^{\infty} a_n (n+\tfrac{3}{2})(n+\tfrac{1}{2}) x^{n-\frac{1}{2}}.$$

Write $x^2 y$ as $\sum_{n=2}^{\infty} a_{n-2} x^{n+\frac{3}{2}}$ and plug all these into equation (1):

$$a_0 \cdot 0 \cdot x^{\frac{3}{2}} + 2a_1 \cdot x^{\frac{5}{2}} + \sum_{n=2}^{\infty} \left[n(n+1) a_n + a_{n-2} \right] x^{n+\frac{3}{2}} = 0$$

Hence a_0 can be any constant, $a_1 = 0$ and

$$n(n+1) a_n + a_{n-2} = 0 \quad \text{for } n = 2, 3, \cdots$$

This recursion formula implies for $n = 0, 1, 2, 3, \cdots$

$a_{2n+1} = 0$, while $a_{2n} = \dfrac{-a_{2n-2}}{(2n+1)\cdot 2n} = \dfrac{a_{2n-4}}{(2n+1)2n(2n-1)(2n-2)}$

$$= \cdots = \frac{(-1)^n a_0}{(2n+1)!}$$

Therefore $$y = x^{\frac{3}{2}} a_0 \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n+1)!} = a_0 x^{\frac{1}{2}} \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

$$= a_0 x^{\frac{1}{2}} \sin x$$

One solution of (1) is $x^{\frac{1}{2}} \sin x$.

4-2

Prove that the Laplace transform formula $\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}$ is valid for all positive integers.

****

For $n=1$ we have by definition or evaluation of integral which defines the Laplace transform

$$\mathcal{L}\{t\} = \int_0^\infty t e^{-st} dt$$

Integration by parts yields

$$\mathcal{L}\{t\} = \frac{1}{s^2} \qquad \text{note: } \frac{1!}{s^2} = \frac{1}{s^2}$$

Assume $\mathcal{L}\{t^k\} = \frac{k!}{s^{k+1}}$ for k a positive integer.

Then $\mathcal{L}\{t^{k+1}\} = \int_0^\infty e^{-st} t^{k+1} dt$

Integration by parts yields

Let $u = t^{k+1}$, $dv = e^{-st} dt$

$du = (k+1) t^k dt$, $v = -\frac{1}{s} e^{-st}$

and $\int_0^\infty e^{-st} t^{k+1} dt = \lim_{n\to\infty} \left[-\frac{1}{s} t^{k+1} e^{-st}\right]_0^n + \frac{k+1}{s} \int_0^\infty e^{-st} t^k dt$

$$= \frac{k+1}{s} \frac{k!}{s^{k+1}}$$

$$= \frac{(k+1)!}{s^{k+2}}$$

By the induction principle the formula is established.

4
THE LAPLACE TRANSFORM

FORWARD TRANSFORM OF ELEMENTARY FUNCTIONS

4-1

Use $\cos^2 A = 1/2 + 1/2 \cos 2A$ to find $L\left\{\cos^2 2t\right\}$ from simpler transforms.

****

$$L\left\{\cos^2 2t\right\} = L\left\{\frac{1}{2}\right\} + L\left\{\frac{1}{2}\cos 4t\right\}$$

$$= \frac{1}{2}L\{1\} + \frac{1}{2}L\{\cos 4t\}$$

$$= \frac{1}{2}\left(\frac{1}{s} + \frac{s}{s^2+16}\right)$$

$$= \frac{1}{2}\left(\frac{2s^2+16}{s(s^2+16)}\right) = \frac{s^2+8}{s(s^2+16)}$$

4-4

Find the Laplace transform of the following functions:

(i) t cosh at

(ii) e^{at} sin bt

(iii) t - t $u_1(t)$

(iv) $\int_0^t (t-r) \sin r \, dr$.

****

(i) We use the formula: $\mathcal{L}\{\cosh at\} = \frac{s}{s^2-a^2}$

and $\mathcal{L}\{t f(t)\} = -\frac{d}{ds}F(s)$. Therefore

$$\mathcal{L}\{t \cosh at\} = -\frac{d}{ds}\left[\frac{s}{s^2-a^2}\right] = \frac{s^2+a^2}{(s^2-a^2)^2}.$$

(ii) From table : $\mathcal{L}\{\sin bt\} = \frac{b}{s^2+b^2}$

and use $\mathcal{L}\{e^{at} f(t)\} = F(s-a)$. Hence

$$\mathcal{L}\{e^{at} \sin bt\} = \frac{b}{(s-a)^2+b^2}.$$

(iii) Rewrite $t - t u_1(t) = t - (t-1) u_1(t) - u_1(t)$

and use $\mathcal{L}\{u_c(t) f(t-c)\} = e^{-cs} F(s)$ and $\mathcal{L}\{u_c(t)\} = \frac{e^{-cs}}{s}$.

These give

$$\mathcal{L}\{t - t u_1(t)\} = \frac{1}{s^2} - \frac{e^{-s}}{s^2} - \frac{e^{-s}}{s}.$$

(iv) From the convolution formula

$$\mathcal{L}\left\{\int_0^t f(r) g(t-r) \, dr\right\} = \mathcal{L}\{f(t)\} \mathcal{L}\{g(t)\}$$

we have

$$\mathcal{L}\left\{\int_0^t (t-r) \sin r \, dr\right\} = \mathcal{L}\{t\} \mathcal{L}\{\sin t\} = \frac{1}{s^2} \cdot \frac{1}{s^2+1}.$$

4-3

Use the definition of the Laplace Transform to find $\mathcal{L}\{f(t)\}$ where $f(t) = t + \sin 2t - e^{-2t}$

****

By definition, $\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt$.

$$= \int_0^\infty e^{-st}(t + \sin 2t - e^{-2t})\, dt$$

$$= \int_0^\infty te^{-st} dt + \int_0^\infty e^{-st} \sin 2t\, dt - \int_0^\infty e^{-(s+2)t} dt$$

Now, $\int_0^\infty te^{-st} dt = \frac{-te^{-st}}{s}\Big|_0^\infty + \frac{1}{s}\int_0^\infty e^{-st} dt = \frac{1}{s^2}$

$$\int_0^\infty e^{-st} \sin 2t\, dt = \frac{-e^{-st}\sin 2t}{s}\Big|_0^\infty + \frac{2}{s}\int_0^\infty e^{-st}\cos 2t\, dt$$

$$= \frac{2}{s}\left[\frac{-e^{-st}\cos 2t}{s}\Big|_0^\infty - \frac{2}{s}\int_0^\infty e^{-st}\sin 2t\, dt\right]$$

$$= \frac{2}{s^2} - \frac{4}{s^2}\int_0^\infty e^{-st}\sin 2t\, dt$$

Which implies that $\int_0^\infty e^{-st}\sin 2t\, dt = \frac{2}{s^2+4}$

and $\int_0^\infty e^{-(s+2)t} dt = \frac{-e^{-(s+2)t}}{s+2}\Big|_0^\infty = \frac{1}{s+2}$

Thus, $\mathcal{L}\{f(t)\} = \frac{1}{s^2} + \frac{2}{s^2+4} - \frac{1}{s+2}$.

4-6

Use integration by parts to derive $\mathcal{L}\{te^{at}\}$. For which values of s is this valid?

****

By definition, $\mathcal{L}\{te^{at}\} = \int_0^\infty te^{at}e^{-st}\,dt = \int_0^\infty te^{-t(s-a)}\,dt$

Integrate by parts with $u = t$, $dv = e^{-t(s-a)}\,dt$; $du = dt$, $v = \dfrac{e^{-t(s-a)}}{-(s-a)}$

$$\mathcal{L}\{te^{at}\} = t\,\frac{e^{-t(s-a)}}{-(s-a)}\Big|_{t=0}^{t=\infty} - \int_0^\infty \frac{e^{-t(s-a)}}{-(s-a)}\,dt$$

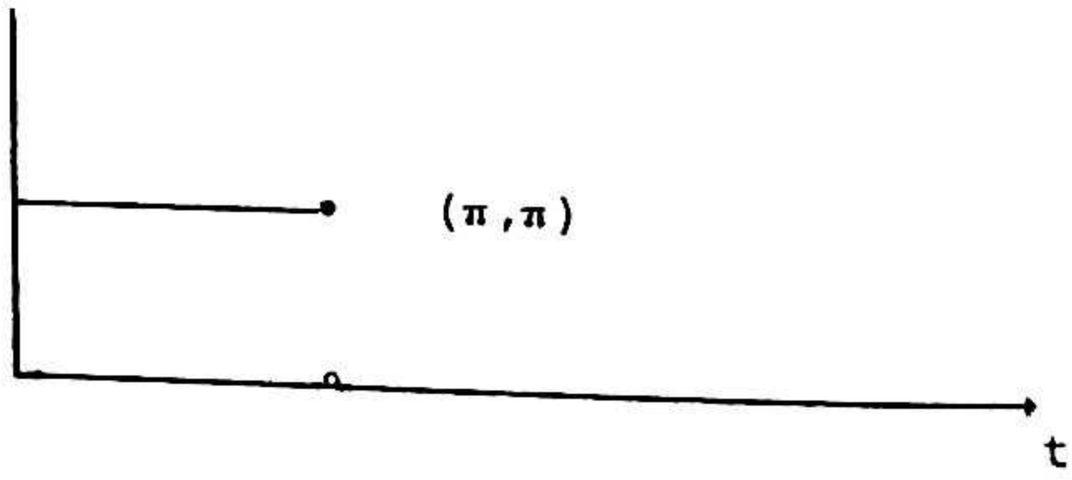
if $s-a>0$, then $\lim_{t\to\infty} te^{-t(s-a)} = 0$ and

$$\mathcal{L}\{te^{at}\} = 0 + \frac{1}{s-a}\int_0^\infty e^{-t(s-a)}\,dt = -\frac{1}{(s-a)^2}\,e^{-t(s-a)}\Big|_{t=0}^{t=\infty}$$

$$\mathcal{L}\{te^{at}\} = \frac{1}{(s-a)^2} \qquad \text{if } s>a\,.$$

4-7

Find the Laplace transform of f(t) given in the figure



****

From the graph $f(t) = \begin{cases} \pi, & 0 \le t \le \pi \\ 0, & t > \pi \end{cases}$

4-5

Use the definition of the Laplace Transform of to compute $\mathcal{L}\{e^{3t}\}$.

****

The definition of the Laplace Transform of $f(t)$ is $\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt$.

Here we have $f(t) = e^{3t}$

$$\mathcal{L}\{e^{3t}\} = \int_0^\infty e^{-st} e^{3t}\,dt = \int_0^\infty e^{-(s-3)t}\,dt$$

We need $\int e^{-(s-3)t}\,dt$ Let $u = -(s-3)t$ Then $du = -(s-3)\,dt$

Substituting we get

$$\int e^{-(s-3)t}\,dt = -\frac{1}{s-3}\int -(s-3)\,e^{-(s-3)t}\,dt$$

$$= -\frac{1}{s-3}\int e^u\,du = -\frac{1}{s-3}\,e^u + C = -\frac{1}{s-3}e^{-(s-3)t} + C$$

Using the Fundamental Theorem of Calculus gives

$$\int_0^\infty e^{-(s-3)t}\,dt = \lim_{b\to\infty}\left[-\frac{1}{s-3}\,e^{-(s-3)t}\right]\Big|_0^b$$

$$= \lim_{b\to\infty}\left\{\left[-\frac{1}{s-3}\,e^{-(s-3)b}\right] - \left[-\frac{1}{s-3}\right]\right\}$$

$$= \frac{1}{s-3} \quad \text{for} \quad s > 3.$$

$$= \frac{e^{-2s} - e^{-s} + 1}{s}$$

4-9

Determine the Laplace transform of the function defined as

$$F(t) = \begin{cases} t, & 0 < t < 25 \\ 0, & t > 25 \end{cases}$$

****

By definition the Laplace transform of F(t) is given as

$$\mathcal{L}\{F(t)\} = \int_0^\infty e^{-st} F(t)\, dt$$

$$= \int_0^{25} t e^{-st} dt + \int_{25}^{0} 0 \cdot e^{-st} dt$$

$$= \int_0^{25} t e^{-st} dt$$

Integration by parts yields

$$\int_0^{25} t e^{-st} dt = \left[-\frac{t}{s} e^{-st}\right]_0^{25} + \int_0^{25} \frac{1}{s} e^{-st} dt$$

$$= -\frac{25}{s} e^{-25s} + \frac{1}{s}\left(-\frac{1}{s} e^{-st}\right)\Big|_0^{25}$$

$$= -\frac{25}{s} e^{-25s} - \frac{1}{s^2} e^{-25s} + \frac{1}{s^2}$$

$$= \underline{\underline{\frac{1}{s^2}\left(1 - e^{-25s} - 25 e^{-25s}\right)}}$$

$$\mathcal{L}\{f(t)\} = F(s) = \int_0^\infty e^{-st} f(t)\,dt$$

$$= \int_0^\pi e^{-st}\,\pi\,dt + \int_\pi^\infty e^{-st}\cdot 0\,dt$$

$$= \left[-\frac{\pi}{s}e^{-st}\right]_{t=0}^{\pi} + 0$$

$$\boxed{F(s) = \frac{\pi}{s}(1-e^{-s\pi})}$$

4-8

Use the definition of the Laplace Transform to find the Laplace Transform of the function

$$f(t) = \begin{cases} 1 & 0 \le t < 1 \\ 0 & 1 \le t < 2 \\ 1 & 2 \le t \end{cases}$$

****

$$\mathcal{L}(f(t)) = \int_0^\infty e^{-st} f(t)\,dt$$

$$= \int_0^1 e^{-st}\,dt + \int_2^\infty e^{-st}\,dt$$

$$= \frac{e^{-st}}{-s}\Big|_0^1 + \lim_{B\to\infty} \frac{e^{-st}}{-s}\Big|_2^B$$

$$= \frac{e^{-s}}{-s} - \left(\frac{1}{-s}\right) + \lim_{B\to\infty} \frac{e^{-sB}}{-s} - \left(\frac{e^{-2s}}{-s}\right)$$

$$= -\frac{e^{-s}}{s} + \frac{1}{s} + \frac{e^{-2s}}{s}$$

4-11

Find the Laplace transform for each of the following functions:

a) $f(t) = 3t^2 - 8t + 7$ b) $f(t) = t^5 e^{2t}$ c) $f(t) = e^{-t}\cos 3t$

****

a) $\mathcal{L}(t^n) = \frac{n!}{s^{n+1}}$ $n = 0, 1, 2, 3, \cdots$

$$\Rightarrow \mathcal{L}(3t^2 - 8t + 7) = 3\mathcal{L}(t^2) - 8\mathcal{L}(t^1) + 7\mathcal{L}(t^0)$$

$$= 3\left(\frac{2}{s^3}\right) - 8\left(\frac{1}{s^2}\right) + 7\left(\frac{1}{s}\right)$$

$$= \frac{6}{s^3} - \frac{8}{s^2} + \frac{7}{s}$$

$$= \frac{6 - 8s + 7s^2}{s^3}$$

b) $\mathcal{L}(t^n e^{at}) = \frac{n!}{(s-a)^{n+1}}$ $n = 0, 1, 2, 3, \cdots$

$$\Rightarrow \mathcal{L}(t^5 e^{2t}) = \frac{5!}{(s-2)^6} = \frac{120}{(s-2)^6}$$

c) $\mathcal{L}(e^{at}\cos bt) = \frac{s-a}{(s-a)^2 + b^2}$

$\mathcal{L}(e^{at}\sin bt) = \frac{b}{(s-a)^2 + b^2}$

$$\Rightarrow \mathcal{L}(e^{-t}\cos 3t) = \frac{s+1}{(s+1)^2 + 3^2} = \frac{s+1}{(s+1)^2 + 9}$$

$$= \frac{s+1}{s^2 + 2s + 10}$$

4-10

Find the Laplace transform of the following functions:

i) $t \cos 2t$, ii) $t^2 e^{3t} + t \sin 3t$

****

i) $L\{\cos 2t\} = \frac{s}{s^2+4}$

$L\{(-t)\cos 2t\} = \frac{d}{ds}\left(\frac{s}{s^2+4}\right) = \frac{4-s^2}{(s^2+4)^2}$

Therefore $L\{t \cos 2t\} = \frac{s^2-4}{(s^2+4)^2}$

ii) $L\{e^{3t}\} = \frac{1}{s-3}$

$L\{(-t)e^{3t}\} = \frac{d}{ds}\left(\frac{1}{s-3}\right) = -\frac{1}{(s-3)^2}$

$L\{(-t)(-t)e^{3t}\} = \frac{d}{ds}\left(-\frac{1}{(s-3)^2}\right) = \frac{2}{(s-3)^3}$

that is $L\{t^2 e^{3t}\} = \frac{2}{(s-3)^3}$

Also $L\{\sin 3t\} = \frac{3}{s^2+9}$

$L\{(-t)\sin 3t\} = \frac{d}{ds}\left(\frac{3}{s^2+9}\right) = \frac{-6s}{(s^2+9)^2}$

that is, $L\{t \sin 3t\} = \frac{6s}{(s^2+9)^2}$

By linearity of L-transform

$L\{t^2 e^{3t} + t \sin 3t\} = L\{t^2 e^{3t}\} + L\{t \sin 3t\}$

$= \frac{2}{(s-3)^3} + \frac{6s}{(s^2+9)^2}$

4-13

Compute the inverse Laplace Transform of

$$\frac{s^2 - 2s - 1}{s^2(s-1)} \quad \text{-i.e.} \quad \mathcal{L}^{-1}\left\{\frac{s^2 - 2s - 1}{s^2(s-1)}\right\}.$$

****

Using partial fractions we have

$$\frac{s^2-2s-1}{s^2(s-1)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s-1} \quad \text{or}$$

$$s^2-2s-1 = As(s-1) + B(s-1) + Cs^2.$$

$$s^2-2s-1 = As^2 - As + Bs - B + Cs^2$$

$$s^2-2s-1 = (A+C)s^2 + (-A+B)s + (-B)$$

Equating coefficients of like powers of s we get

$$\begin{aligned} A \quad +C &= 1 \\ -A+B &= -2 \\ -B &= -1 \end{aligned}$$

Solving this system by substitution gives

$$A = 3, \; B = 1, \; C = -2$$

So $$\frac{s^2-2s-1}{s^2(s-1)} = \frac{3}{s} + \frac{1}{s^2} - \frac{2}{s-1}$$

Using the linearity of $\mathcal{L}^{-1}$ we get

$$\mathcal{L}^{-1}\left\{\frac{s^2-2s-1}{s^2(s-1)}\right\} = \mathcal{L}^{-1}\left\{\frac{3}{s} + \frac{1}{s^2} - \frac{2}{s-1}\right\}$$

$$= 3\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} - 2\mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\}$$

$$= 3 \cdot 1 + t - 2 \cdot e^t \quad \text{by table look-up}$$

$$= 3 + t - 2e^t.$$

INVERSE TRANSFORM OF SELECTED FUNCTIONS

4-12

Find the inverse Laplace transform of:

i) $\frac{3s+1}{s^3}$, ii) $\frac{5s+3}{(s^2+4)}$, iii) $\frac{4}{s(s^2+1)}$

****

i) $\mathcal{L}^{-1}\left[\frac{3s+1}{s^3}\right] = \mathcal{L}^{-1}\left[\frac{3s}{s^3}\right] + \mathcal{L}^{-1}\left[\frac{1}{s^3}\right]$

$= 3\,\mathcal{L}^{-1}\left[\frac{1}{s^2}\right] + \mathcal{L}^{-1}\left[\frac{1}{s^3}\right]$

$= 3t + \frac{1}{2}t^2$ $\qquad \mathcal{L}^{-1}\left[\frac{1}{s^{n+1}}\right] = \frac{t^n}{n!}$

ii) $\mathcal{L}^{-1}\left[\frac{5s+3}{s^2+4}\right] = 5\,\mathcal{L}^{-1}\left[\frac{s}{s^2+4}\right] + \frac{3}{2}\mathcal{L}^{-1}\left[\frac{2}{s^2+4}\right]$

(linearity of inverse Laplace transform)

$= 5\cos 2t + \frac{3}{2}\sin 2t$.

iii) $\mathcal{L}^{-1}\left[\frac{4}{s(s^2+1)}\right] = 4\,\mathcal{L}^{-1}\left[\frac{1}{s(s^2+1)}\right]$

$= 4\mathcal{L}^{-1}\left[\frac{1}{s} - \frac{s}{s^2+1}\right]$

$= 4\left[\mathcal{L}^{-1}\left[\frac{1}{s}\right] - \mathcal{L}^{-1}\left[\frac{s}{s^2+1}\right]\right]$

$= 4(1 - \cos t)$

4-15

Given that the Laplace transform of a function f(t) is

$f(s) = \dfrac{8s - 12}{s^2 + 6s - 27}$, determine f(t).

****

$f(t) = \mathcal{L}^{-1}\{f(s)\}$

By partial fractions we have

$$\frac{8s+12}{s^2+6s-27} = \frac{A}{s+9} + \frac{B}{s-3}$$

$$\frac{8s+12}{s^2+6s-27} = \frac{A(s-3)+B(s+9)}{s^2+6s-27}$$

or

$$\frac{8s+12}{s^2+6s-27} = \frac{(A+B)s + 9B - 3A}{s^2+6s-27}$$

Equating coefficients of similar terms we obtain

1) $A + B = 8$
2) $9B - 3A = 12$

this set of equations yields $A = 5$, $B = 3$.

Consequently $\mathcal{L}^{-1}\left\{\frac{8s+12}{s^2+6s-27}\right\} = \mathcal{L}^{-1}\left\{\frac{5}{s+9}\right\} + \mathcal{L}^{-1}\left\{\frac{3}{s-3}\right\}$

and $\mathcal{L}^{-1}\{f(s)\} = f(t) = \underline{\underline{5e^{-9t} + 3e^{3t}}}$

4-14

Find the inverse Laplace transform of $F(s) = \frac{2s+1}{s^3+1}$.

****

$$F(s) = \frac{2s+1}{(s+1)(s^2-s+1)}$$

Use partial fractions:

$$\frac{2s+1}{(s+1)(s^2-s+1)} \equiv \frac{A}{s+1} + \frac{Bs+C}{s^2-s+1}$$

$$2s+1 \equiv A(s^2-s+1) + (Bs+C)(s+1) \quad \text{— (1)}$$

$$2s+1 \equiv (A+B)s^2 + (B+C-A)s + (A+C) \quad \text{— (2)}$$

$s=-1$ in (1) $\Longrightarrow$ $-1 = 3A$, i.e. $A = -\frac{1}{3}$

From (2) $A+B=0 \Longrightarrow B = \frac{1}{3}$ and $A+C=1 \Longrightarrow C = \frac{4}{3}$

$$\therefore F(s) = \frac{\frac{1}{3}s + \frac{4}{3}}{s^2-s+1} - \frac{1}{3}\cdot\frac{1}{s+1}$$

$$= \frac{\frac{1}{3}s + \frac{4}{3}}{(s-\frac{1}{2})^2 + (\frac{\sqrt{3}}{2})^2} - \frac{1}{3}\cdot\frac{1}{s+1}$$

$$= \frac{1}{3}\left[\frac{(s-\frac{1}{2}) + \frac{9}{2}}{(s-\frac{1}{2})^2 + (\frac{\sqrt{3}}{2})^2}\right] - \frac{1}{3}\cdot\frac{1}{s+1}$$

$$= \frac{1}{3}\cdot\frac{s-\frac{1}{2}}{(s-\frac{1}{2})^2 + (\frac{\sqrt{3}}{2})^2} + \sqrt{3}\cdot\frac{\frac{\sqrt{3}}{2}}{(s-\frac{1}{2})^2 + (\frac{\sqrt{3}}{2})^2} - \frac{1}{3}\cdot\frac{1}{s+1}$$

$$\mathcal{L}^{-1}\{F\}(t) = \frac{1}{3}e^{\frac{1}{2}t}\cos\frac{\sqrt{3}}{2}t + \sqrt{3}\,e^{\frac{1}{2}t}\sin\frac{\sqrt{3}}{2}t - \frac{1}{3}e^{-t}.$$

4-17

Find the inverse Laplace transform of the function

$$F(s) = \frac{3s + 5}{s(s^2 + 4s + 5)} .$$

* *

First we simplify the given function using the method of partial fractions.

Let $$\frac{3S+5}{S(S^2+4S+5)} = \frac{A}{S} + \frac{BS+C}{S^2+4S+5} .$$

Multiplying both sides of this equation by the common denominator $S(S^2+4S+5)$ we have

$$A(S^2+4S+5) + S(BS+C) = 3S+5.$$

Next we equate terms containing like powers in S to obtain

$$S^0: \quad 5A = 5 \quad \Rightarrow \quad A = 1$$

$$S^1: \quad 4A + C = 3 \quad \Rightarrow \quad C = -1$$

$$S^2: \quad A + B = 0 \quad \Rightarrow \quad B = -1 .$$

Therefore

$$F(S) = \frac{1}{S} - \frac{S+1}{S^2+4S+5}$$

which may be written as

$$F(S) = \frac{1}{S} - \frac{(S+2)-1}{(S+2)^2+1}$$

4-16

Find the inverse Laplace transform for each of the following functions:

a) $F(s) = \dfrac{3s + 1}{s^3 - 2s^2 + s}$ b) $F(s) = \dfrac{2s}{s^2 - 6s + 10}$

****

a) PARTIAL FRACTIONS : $\dfrac{3s+1}{s(s-1)^2} = \dfrac{A}{s} + \dfrac{B}{s-1} + \dfrac{C}{(s-1)^2}$

$$3s+1 = A(s-1)^2 + Bs(s-1) + Cs$$

$s=0 \Rightarrow 1 = A$

$s=1 \Rightarrow 4 = C$

$s=2 \Rightarrow 7 = A + 2B + 2C = 1 + 2B + 8$

$-2 = 2B$

$B = -1$

$$\mathcal{L}^{-1}\left(\frac{3s+1}{s^3-2s^2+s}\right) = \mathcal{L}^{-1}\left(\frac{1}{s} - \frac{1}{s-1} + \frac{4}{(s-1)^2}\right)$$

$$= 1 - e^t + 4te^t$$

b) COMPLETE THE SQUARE AND ADJUST THE NUMERATOR :

$$\frac{2s}{s^2-6s+10} = \frac{2s}{s^2-6s+9+1} = \frac{2s}{(s-3)^2+1}$$

$$= \frac{2s-6+6}{(s-3)^2+1}$$

$$= \frac{2(s-3)}{(s-3)^2+1} + \frac{6}{(s-3)^2+1}$$

$$\mathcal{L}^{-1}\left(\frac{2s}{s^2-6s+10}\right) = \mathcal{L}^{-1}\left(\frac{2(s-3)}{(s-3)^2+1} + \frac{6}{(s-3)^2+1}\right)$$

$$= 2e^{3t}\cos t + 6e^{3t}\sin t$$

$$= (2\cos t + 6\sin t)e^{3t}$$

4-18

Find the inverse LaPlace transform of $f(s) = \dfrac{S+2}{S^2-6S+8}$

****

$$\frac{S+2}{S^2-6S+8} = \frac{S+2}{(S-2)(S-4)} = \frac{A}{S-2} + \frac{B}{S-4}$$

$$S+2 = A(S-4) + B(S-2)$$
$$S+2 = (A+B)S + (-4A-2B)$$

So $\left.\begin{aligned} A+B &= 1 \\ -4A-2B &= 2 \end{aligned}\right\}$

Solving this system, $A = -2$, $B = 3$

$$L^{-1}\left\{\frac{S+2}{S^2-6S+8}\right\} = L^{-1}\left\{\frac{3}{S-4} - \frac{2}{S-2}\right\}$$
$$= 3e^{4t} - 2e^{2t}$$

SHIFTING ON THE S-AXIS

4-19

Evaluate each Laplace or inverse Laplace transform.

a) $\mathcal{L}\{(1-e^t)^2\}$

b) $\mathcal{L}^{-1}\left\{\dfrac{4}{(s^2+1)^2}\right\}$

c) $\mathcal{L}^{-1}\left\{\dfrac{1}{s^2+2s+5}\right\}$

****

$$= \frac{1}{s} - \frac{s+2}{(s+2)^2+1} + \frac{1}{(s+2)^2+1}. \qquad (*)$$

We now invoke the translation theorem, namely,

$$e^{at} g(t) = \mathcal{L}^{-1}\{G(s-a)\} = \mathcal{L}^{-1}\left\{ G(s)\Big|_{s \to (s-a)} \right\}$$

and the results

$$\mathcal{L}^{-1}\left\{ \frac{s}{s^2+\omega^2} \right\} = \cos \omega t,$$

$$\mathcal{L}^{-1}\left\{ \frac{\omega}{s^2+\omega^2} \right\} = \sin \omega t.$$

(Consult a table of Laplace transforms.)

Then it follows from (*) that

$$\mathcal{L}^{-1}\{F(s)\} = 1 - e^{-2t}\cos t + e^{-2t}\sin t$$

$$= 1 + e^{-2t}(\sin t - \cos t).$$

[Note: $\mathcal{L}^{-1}\left\{ \frac{s+2}{(s+2)^2+1} \right\} = e^{-2t}\cos t$

and

$$\mathcal{L}^{-1}\left\{ \frac{1}{(s+2)^2+1} \right\} = e^{-2t}\sin t\,].$$

4-21

Find the inverse Laplace transform: $\mathcal{L}^{-1}\left\{\frac{4s}{s^2 + 6s + 10}\right\}$

****

$\mathcal{L}^{-1}\left\{\frac{4s}{s^2+6s+10}\right\}$ note that the denominator doesn't factor so convolution or partial fraction decomposition is not appropriate.

$$\frac{4s}{s^2+6s+10} = \frac{4s}{s^2+6s+9+1} = \frac{4s}{(s+3)^2+1}$$

Since there is an $s+3$ in the denominator there must also be in the numerator before taking the inverse Laplace transform

$$= \frac{4(s+3)}{(s+3)^2+1} - \frac{12}{(s+3)^2+1}$$

$$\text{so } \mathcal{L}^{-1}\left\{\frac{4(s+3)}{(s+3)^2+1} - \frac{12}{(s+3)^2+1}\right\} =$$

$$\underline{\underline{4e^{-3t}\cos t - 12e^{-3t}\sin t}}$$

4-22

Compute the Laplace Transform of $e^{3t}\cos 2t$ -i.e. $\mathcal{L}\left\{e^{3t}\cos 2t\right\}$.

****

We use the formula for shifting along the s-axis.

$$\mathcal{L}\left\{e^{at}f(t)\right\} = F(s-a) \quad \text{for } s > a+b$$

where $F(s) = \mathcal{L}\left\{f(t)\right\}$.

a) $\mathcal{L}\{(1-e^t)^2\} = \mathcal{L}\{1-2e^t+e^{2t}\} = \mathcal{L}\{1\} - 2\mathcal{L}\{e^t\} + \mathcal{L}\{e^{2t}\}$

$= \frac{1}{s} - \frac{2}{s-1} + \frac{1}{s-2}$

b) $\mathcal{L}^{-1}\left\{\frac{4}{(s^2+1)^2}\right\} = 2\mathcal{L}^{-1}\left\{\frac{2}{(s^2+1)^2}\right\} = 2\sin t - 2t\cos t$

c) $\mathcal{L}^{-1}\left\{\frac{1}{s^2+2s+5}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2+4}\right\} = \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{2}{(s+1)^2+4}\right\}$

$= \frac{1}{2}e^{-t}\sin 2t$

4-20

Determine the inverse Laplace transform of

$$Y(s) = \frac{s+4}{s^2+4s+13}.$$

****

$Y(s) = \frac{s+4}{s^2+4s+13} = \frac{s+4}{(s+2)^2+9} = \frac{(s+2)+2}{(s+2)^2+3^2}$

$y(t) = \mathcal{L}^{-1}\left\{\frac{s+2}{(s+2)^2+3^2}\right\} + \mathcal{L}^{-1}\left\{\frac{2}{(s+2)^2+3^2}\right\}$

By the shift theorem on the s-axis,

$\mathcal{L}^{-1}\{F(s-a)\} = e^{at}f(t)$

and since $\mathcal{L}^{-1}\left\{\frac{s}{s^2+k^2}\right\} = \cos kt$, $\mathcal{L}^{-1}\left\{\frac{k}{s^2+k^2}\right\} = \sin kt$

$$\boxed{y(t) = e^{-2t}\cos 3t + \frac{2}{3}e^{-2t}\sin 3t}$$

4-24

$y'' + y' = e^{-t}\sin t \qquad y(0) = y'(0) = 0.$

$Y(s) =$ (A) $1/s(s+1)^3$

(B) $1/s(s+1)(s^2+1)$

(C) $s/(s+1)[(s+1)^2+1]$

(D) $1/s(s+1)[(s+1)^2+1]$

(E) $1/s[(s+1)^2+1]$

****

Multiplication by e^{-t} translates the transform to $s-(-1) = s+1$. Since $\sin t \to \frac{1}{s^2+1}$ we obtain, $e^{-t}\sin t \to \frac{1}{(s+1)^2+1}$.

The derivatives y'' and y' transform by

$$y'' \to s^2Y - s\underbrace{y'(0)}_{0} - \underbrace{y(0)}_{0} = s^2Y.$$

$$y' \to sY - \underbrace{y(0)}_{0} = sY.$$

The transformed equation is

$$s^2Y + sY = \frac{1}{(s+1)^2+1}.$$

That is

$$(s^2+s)Y = \frac{1}{(s+1)^2+1}.$$

Hence

$$Y = \frac{1}{s(s+1)[(s+1)^2+1]}$$

The correct answer is (D).

Here $f(t) = \cos 2t$ and $a = 3$

Since $\mathcal{L}\{\cos 2t\} = \frac{s}{s^2+4}$,

we have $\mathcal{L}\{e^{3t}\cos 2t\} = \frac{s-3}{(s-3)^2+4}$.

4-23

Show that for n an non-negative integer that the inverse Laplace transform has the property

$$\mathcal{L}^{-1}\left\{\frac{1}{(s+2)^{n+1}}\right\} = \frac{e^{-2t}t^n}{n!}$$

****

The shifting property of Laplace transforms yields

$$\mathcal{L}\{e^{at}F(t)\} = f(s-a)$$

therefore $\mathcal{L}^{-1}\{f(s-a)\} = e^{at}F(t)$

and $\mathcal{L}^{-1}\{f(s+a)\} = \mathcal{L}^{-1}\{f(s-(-a))\}$

$= e^{-at}F(t)$

Note: $\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}$, $s>0$, $n=1, 2, 3, \ldots$

Consider $F(t) = t^n$, then $\mathcal{L}\{e^{-2t}t^n\} = \frac{n!}{(s+2)^{n+1}}$

and thus $e^{-2t}t^n = \mathcal{L}^{-1}\left\{\frac{1}{(s+2)^{n+1}}\right\}$

4-27

Let $h(t) = t^3 \sinh 5t$. Compute $H(s) = L\{h(t)\}$, the Laplace Transform of h. [Comment: there are several ways to solve this problem; at least one of the ways is very short and easy; some of the other methods can be rather long.]

****

1. $\mathcal{L}\{h\} = \int_0^\infty e^{-st} t^3 \sinh 5t \, dt$ (original definition) followed by several integrations-by-parts.

2. $\mathcal{L}\{t^3 \sinh 5t\} = -\mathcal{L}\{(-t)^3 \sinh 5t\}$

$= -(\mathcal{L}(\sinh 5t))''' = -\left(\frac{5}{s^2-25}\right)''' = \cdots$.

3. $\sinh 5t = \frac{1}{2}(e^{5t} - e^{-5t})$.

$\mathcal{L}\{t^3 \sinh 5t\} = \frac{1}{2}(\mathcal{L}\{t^3 e^{5t}\} - \mathcal{L}\{t^3 e^{-5t}\})$

$= \frac{1}{2}(-\mathcal{L}\{(-t)^3 e^{5t}\} + \mathcal{L}\{(-t)^3 e^{-5t}\})$

$= -\frac{1}{2}(\mathcal{L}\{e^{5t}\})''' + \frac{1}{2}(\mathcal{L}\{e^{-5t}\})'''$

$= -\frac{1}{2}\left(\frac{1}{s-5}\right)''' + \frac{1}{2}\left(\frac{1}{s+5}\right)''' = -\frac{1}{2}\frac{-6}{(s-5)^4} + \frac{1}{2}\frac{-6}{(s+5)^4}$

$= 3\left(\frac{1}{(s-5)^4} - \frac{1}{(s+5)^4}\right)$.

4. $\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}$; $\mathcal{L}\{e^{ct} f(t)\} = F(s-c)$.

$\mathcal{L}\{t^3 \sinh 5t\} = \mathcal{L}\{\frac{1}{2} e^{5t} t^3\} - \mathcal{L}\{\frac{1}{2} e^{-5t} t^3\}$

$= \frac{1}{2}\frac{3!}{(s-5)^4} - \frac{1}{2}\frac{3!}{(s+5)^4} = 3\left(\frac{1}{(s-5)^4} - \frac{1}{(s+5)^4}\right)$.

4-25

Find $L\left[e^{-at}\sin^2 bt\right]$

****

Using the trigonometric identity $\sin^2\theta = \frac{1}{2}(1-\cos 2\theta)$, we have

$$L\left[e^{-at}\sin^2 bt\right] = \frac{1}{2}L\left[e^{-at} - e^{-at}\cos 2bt\right]$$

$$= \frac{1}{2}\left[\frac{1}{s+a} - \frac{s+a}{(s+a)^2 + (2b)^2}\right]$$

$$= \frac{2b^2}{(s+a)\left[(s+a)^2 + 4b^2\right]}.$$

4-26

Find the inverse Laplace transform : $\mathcal{L}^{-1}\left\{\frac{6s^2 + 3s + 45}{(s^2+9)(s-3)}\right\}$

****

$$\mathcal{L}^{-1}\left\{\frac{6s^2+3s+45}{(s^2+9)(s-3)}\right\} = \mathcal{L}^{-1}\left\{\frac{As+B}{s^2+9} + \frac{C}{s-3}\right\}$$

$$6s^2+3s+45 = (As+B)(s-3) + C(s^2+9)$$
$$6s^2+3s+45 = As^2 + Bs - 3As - 3B + Cs^2 + 9C$$

s^2: $6 = A+C$
s : $3 = -3A+B$
const : $45 = -3B+9C$

solving simultaneously $A=0,\ B=3,\ C=6$

$$= \mathcal{L}^{-1}\left\{\frac{3}{s^2+9} + \frac{6}{s-3}\right\} = \underline{\sin 3t + 6e^{3t}}$$

4-29

Find the inverse Laplace transform of each of the following functions:

(a) $G(s) = e^{-2s} \cdot \dfrac{1}{s-1}$

(b) $H(s) = \dfrac{1}{(S-2)^2 + 9}$

(c) $J(s) = e^{-3s} \cdot \dfrac{(S-2)}{(S-2)^2 + 1}$

(d) $K(s) = e^{-s} \cdot \dfrac{S - 3}{S^2 + 3S + 2}$

(e) $M(s) = e^{-3s} \cdot \dfrac{1}{S^2 + 2S + 2}$

****

(a) $g(t) = \mathcal{L}^{-1}G(s) = \mathcal{L}^{-1}\left\{e^{-2s} \cdot \frac{1}{s-1}\right\} = \mathcal{L}^{-1}\left\{e^{-2s} \cdot F(s)\right\}$ where $F(s) = \frac{1}{s-1}$ and $f(t) = e^t$

$\therefore g(t) = u_2(t) f(t-2) = u_2(t)\, e^{(t-2)}$ (where $u_c = \begin{cases} 0 & t<c \\ 1 & t \geq c \end{cases}$)

(b.) $h(t) = \mathcal{L}^{-1}\{H(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{(s-2)^2+9}\right\}$

$= \frac{1}{3}\mathcal{L}^{-1}\left\{\frac{3}{(s-2)^2+9}\right\} = \frac{1}{3}\mathcal{L}^{-1}\{G(s-2)\}$ where $G(s) = \frac{3}{s^2+3^2}$ $\leftrightarrow$ $g(t) = \sin 3t$

$= \frac{1}{3} e^{2t} \sin(3t)$

(c) $j(t) = \mathcal{L}^{-1}\{J(s)\} = \mathcal{L}^{-1}\left\{e^{-3s} \cdot \frac{s-2}{(s-2)^2+1}\right\} \overset{(*1)}{=} \mathcal{L}^{-1}\{e^{-3s} \cdot H(s)\}$

$j(t) \overset{(*1)}{=} u_3(t)\, h(t-3)$ where $H(s)$

$h(t) = \mathcal{L}^{-1}H(s) = \mathcal{L}^{-1}\left\{\frac{s-2}{(s-2)^2+1}\right\}$

$h(t) = \mathcal{L}^{-1}\{G(s-2)\} = e^{2t} \cdot g(t)$ where $G(s) = \frac{s}{s^2+1} \leftrightarrow g(t) = \cos t$

$\therefore h(t) = e^{2t}\cos t$

$\underline{j(t) = u_3(t)\, e^{2(t-3)} \cos(t-3)}$

(d.) $K(t) = \mathcal{L}^{-1}\{K(s)\} = \mathcal{L}^{-1}\left\{e^{-s} \cdot \frac{s-3}{s^2+3s+2}\right\}$; We use partial fractions to rewrite: $\frac{s-3}{s^2+3s+2} = \frac{A}{s+2} + \frac{B}{s+1}$.

4-28

Find the inverse Laplace transform of:

i) $\frac{s}{4s^2+8s+5}$, ii) $\frac{s}{s^2+2s-15}$, iii) $\frac{s}{(s+3)^4}$

****

i) $L^{-1}\left[\frac{s}{4s^2+8s+5}\right] = \frac{1}{4}L^{-1}\left[\frac{s}{s^2+2s+5/4}\right] = \frac{1}{4}L^{-1}\left[\frac{s}{(s+1)^2+(\frac{1}{2})^2}\right]$

$= \frac{1}{4}L^{-1}\left[\frac{s+1-1}{(s+1)^2+(\frac{1}{2})^2}\right] = \frac{1}{4}L^{-1}\left[\frac{s+1}{(s+1)^2+(\frac{1}{2})^2}\right] - \frac{1}{4}L^{-1}\left[\frac{1}{(s+1)^2+(\frac{1}{2})^2}\right]$

$= \frac{1}{4}e^{-t}\cos(\frac{t}{2}) - \frac{1}{4}2\,e^{-t}\sin(\frac{1}{2}t)$

$= \frac{1}{4}e^{-t}\left(\cos\frac{t}{2} - 2\sin\frac{t}{2}\right).$

ii) $L^{-1}\left[\frac{s}{s^2+2s-15}\right] = L^{-1}\left[\frac{3/8}{s-3} + \frac{5/8}{s+5}\right]$

$= \frac{3}{8}L^{-1}\left[\frac{1}{s-3}\right] + \frac{5}{8}L^{-1}\left[\frac{1}{s-(-5)}\right]$

$= \frac{3}{8}e^{3t} + \frac{5}{8}e^{-5t}$

iii) $L^{-1}\left[\frac{s}{(s+3)^4}\right] = L^{-1}\left[\frac{s+3-3}{(s+3)^4}\right] = L^{-1}\left[\frac{s+3}{(s+3)^4}\right] - L^{-1}\left[\frac{3}{(s+3)^4}\right]$

$= L^{-1}\left[\frac{1}{(s+3)^3}\right] - 3L^{-1}\left[\frac{1}{(s+3)^4}\right]$

$= \frac{1}{2!}e^{-3t}t^2 - \frac{3}{3!}e^{-3t}t^3$

$= \frac{1}{2}t^2e^{-3t}(1-t).$

HEAVISIDE FUNCTIONS AND SHIFTING
ON THE T-AXIS
DIRAC'S DELTA FUNCTION
PERIODIC FUNCTIONS

4-30

Find $\mathcal{L}^{-1}\{se^{-s}/(s-1)(s^2+4)\}$ in closed form.

****

First, ignore the e^{-s} and do partial fractions on $\frac{s}{(s-1)(s^2+4)}$

$$\frac{s}{(s-1)(s^2+4)} = \frac{A}{s-1} + \frac{Bs+C}{s^2+4} \quad ; \quad s = A(s^2+4) + (Bs+C)(s-1)$$

With $s=1$, $1 = 5A$, so $A = \frac{1}{5}$. With $s=0$,

$0 = 4A - C$, so $C = 4A = \frac{4}{5}$. Finally with $s=2$,

$2 = 8A + 2B + C$, so $B = 1 - 4A - \frac{C}{2} = -\frac{1}{5}$.

$$\frac{s}{(s-1)(s^2+4)} = \frac{1/5}{s-1} + \frac{-\frac{1}{5}s + 4/5}{s^2+4} = \frac{1}{5}\frac{1}{s-1} - \frac{1}{5}\frac{s}{s^2+4} + \frac{2}{5}\frac{2}{s^2+4}$$

$$\mathcal{L}^{-1}\left\{\frac{s}{(s-1)(s^2+4)}\right\} = \frac{1}{5}e^t - \frac{1}{5}\cos 2t + \frac{2}{5}\sin 2t = f(t)$$

Finally, $\mathcal{L}^{-1}\left\{\frac{se^{-s}}{(s-1)(s^2+4)}\right\} = u_1(t)f(t^*)$, $t^* = t-1$

$$u_1(t) = \begin{cases} 0, & t<1 \\ 1, & t\geq 1 \end{cases}$$

$$= u_1(t)\left[\frac{1}{5}e^{t-1} - \frac{1}{5}\cos 2(t-1) + \frac{2}{5}\sin 2(t-1)\right]$$

$\therefore \quad S-3 = A(s+1) + B(s+2)$

$s^1: \quad 1 = A+B$
$s^0: \quad -3 = A+2B$
$\Rightarrow -4 = B \Rightarrow A = 5; \quad \therefore \frac{S-3}{S^2+3S+2} = \frac{5}{S+2} - \frac{4}{S+1}$

$\therefore \quad K(t) = \mathcal{L}^{-1}\left\{e^{-S}\left[\frac{5}{S+2} - \frac{4}{S+1}\right]\right\}$

$= 5\mathcal{L}^{-1}\left\{e^{-S}\cdot\frac{1}{S+2}\right\} - 4\mathcal{L}^{-1}\left\{e^{-S}\cdot\frac{1}{S+1}\right\}$

$K(t) = 5u_1(t)\,e^{-2(t-1)} - 4u_1(t)\,e^{-(t-1)}$

(e) $m(t) = \mathcal{L}^{-1}\{M(S)\} = \mathcal{L}^{-1}\left\{e^{-3S}\cdot\frac{1}{S^2+2S+2}\right\} = \mathcal{L}^{-1}\{e^{-3S}\cdot F(S)\}$ $(*_1)$

$\overset{(*_1)}{=} u_3(t)\,f(t-3)$ where $F(S) = \frac{1}{S^2+2S+2}$

$m(t) \overset{(*_3)}{=} u_3(t)\,e^{-(t-3)}\sin(t-3)$

$f(t) = \mathcal{L}^{-1}F(s) = \mathcal{L}^{-1}\left\{\frac{1}{(S+1)^2+1}\right\}$

$\overset{*_2}{=} \mathcal{L}^{-1}\{G(S+1)\}$

[where $G(S) = \frac{1}{S^2+1}$ $\updownarrow$ $g(t) = \sin t$]

$\therefore f(t) \overset{(*_2)}{=} e^{-t}g(t) = e^{-t}\sin t$ $(*_3)$

$f(t) = 4 + 2(t-2)u_2(t)$. Now, $\mathcal{L}\{(t-2)u_2(t)\} = \mathcal{L}\{t\}\, e^{-2s}$

$\mathcal{L}\{f(t)\} = \mathcal{L}\{4\} + 2\mathcal{L}\{t\}\, e^{-2s} = \frac{4}{s} + \frac{2}{s^2} e^{-2s}$

4-33

Find $L^{-1}\left[e^{-5s}/(s^2 + 4s + 7)\right]$

****

Completing the square in the denominator, we have $s^2 + 4s + 7 = (s+2)^2 + (\sqrt{3})^2$.

Hence the inverse transform of $1/(s^2+4s+7)$ is $L^{-1}\left[\frac{1}{\sqrt{3}} \cdot \frac{\sqrt{3}}{(s+2)^2 + (\sqrt{3})^2}\right] = \frac{1}{\sqrt{3}} e^{-2t} \sin\sqrt{3}t$.

Now we can obtain the time-delay of 5 units caused by the e^{-5s} numerator. Let $u(t-5)$ be the unit step-function which is zero for $t < 5$ and one for $t \geq 5$. Therefore

$$L^{-1}\left[\frac{e^{-5s}}{s^2+4s+7}\right] = \frac{1}{\sqrt{3}} u(t-5) e^{-2(t-5)} \sin\sqrt{3}(t-5),$$

where t has been replaced by $t-5$ in the inverse transform of $1/(s^2+4s+7)$.

4-31

Find the Laplace transform for the given f(t).

$$f(t) = \begin{cases} t^2 & \text{if } 0 \le t < 1 \\ 0 & \text{if } 1 \le t < 2 \end{cases}$$

where $f(t+2) = f(t)$.

****

We may use $\mathcal{L}(f(t)) = \frac{1}{1-e^{-sT}} \int_0^T e^{-sT} f(t)\, dt$

where T is the period. $T = 2$ in our case.

Thus

$$\mathcal{L}(f(t)) = \frac{1}{1-e^{-2s}} \left[\int_0^1 e^{-st} t^2 dt + \int_1^2 e^{-st} \cdot 0 \cdot dt \right] =$$

$$= \frac{1}{1-e^{-2s}} \left[\left(\frac{t^2 e^{-st}}{-s} - \frac{2te^{-st}}{s^2} - \frac{2e^{-st}}{s^3} \right) \Big|_0^1 + 0 \right] =$$

$$= \frac{1}{1-e^{-2s}} \left[-\frac{2e^{-s}-2}{s^3} - \frac{2e^{-s}}{s^2} - \frac{e^{-s}}{s} \right].$$

4-32

Find the Laplace transform of $f(t) = \begin{cases} 4, & t < 2 \\ 2t, & t \ge 2 \end{cases}$

****

Rewrite f in terms of the step function $u_2(t) = \begin{cases} 0, & t<2 \\ 1, & t \ge 2 \end{cases}$

$$f(t) = 4 + \begin{cases} 0, & t<2 \\ 2t-4, & t \ge 2 \end{cases} = 4 + (2t-4)\, u_2(t)$$

4-35

$\mathcal{L}(|\sin x|) = ?$

****

$$\mathcal{L}(|\sin x|) = \int_0^\infty e^{-px}|\sin x|\,dx = \int_0^\pi e^{-px}|\sin x|\,dx + \int_\pi^{2\pi} + \dots$$

$$= \sum_{j=0}^{\infty}\left(\int_{j\pi}^{(j+1)\pi} e^{-px}|\sin x|\,dx\right).$$ In the jth integral make the change of variables $y = x - j\pi$:

$$\sum_{j=0}^{\infty}\left(\int_0^\pi e^{-p(y+j\pi)}|\sin(y+j\pi)|\,dy\right).$$ $|\sin x|$ is π-periodic (and positive on $(0,\pi)$) so

$$= \sum_{j=0}^{\infty}\left(\int_0^\pi e^{-p(y+j\pi)}\sin y\,dy\right) = \sum_{j=0}^{\infty}\left(e^{-pj\pi}\int_0^\pi e^{-py}\sin y\,dy\right)$$

$$= \left(\sum_{j=0}^{\infty} e^{-pj\pi}\right)\int_0^\pi e^{-py}\sin y\,dy.$$ For $p>0$ the sum is a convergent geometric series,

$$= \frac{1}{1-e^{-p\pi}}.$$ The integral gives:

$$\frac{e^{-py}}{p^2+1}\left(-p\sin y - \cos y\right)\Big|_0^\pi = \frac{e^{-\pi p}+1}{p^2+1}.$$ Thus

$$\boxed{\mathcal{L}(|\sin x|) = \frac{1}{p^2+1}\,\frac{e^{-\pi p}+1}{1-e^{-\pi p}} = \frac{1}{p^2+1}\coth\left(\frac{\pi p}{2}\right)}$$

4-34

Find the Laplace transform of the following functions:

i) $\delta(t - 2\pi)$, ii) $H_2(t)\,(t - 2)$,

iii) $H_3(t)\,(t - 3)e^{4(t-3)}$, iv) $H_4(t)\,(t - 4)\sin 2(t - 4)$

****

i) $L\{\delta(t-2\pi)\} = e^{-2\pi s}$

$L\{\delta(t-t_0)\} = e^{-st_0}$ $(t_0 \geq 0)$

ii) $L\{H_2(t)\,(t-2)\} = e^{-2s}\,L\{t\}$

$= \dfrac{e^{-2s}}{s^2}$

$L\{H_c(t)\,f(t-c)\} = e^{-cs}\,L\{f(t)\}$, where $H_c(t)$ is the Heaviside function

iii) $L\{H_3(t)(t-3)e^{4(t-3)}\}$

$= e^{-3s}\,L\{t\,e^{4t}\}$

$= \dfrac{e^{-3s}}{(s-4)^2}$

iv) $L\{H_4(t)\,(t-4)\sin 2(t-4)\}$

$= e^{-4s}\,L\{t\sin 2t\}$

$= \dfrac{4e^{-4s}\,s}{(s^2+4)^2}$

Now $L\{\sin 2t\} = \dfrac{2}{s^2+4}$

$L\{(-t)\sin 2t\} = \dfrac{d}{ds}\left(\dfrac{2}{s^2+4}\right)$

$= \dfrac{-4s}{(s^2+4)^2}$

that is $L\{t\sin 2t\} = \dfrac{4s}{(s^2+4)^2}$

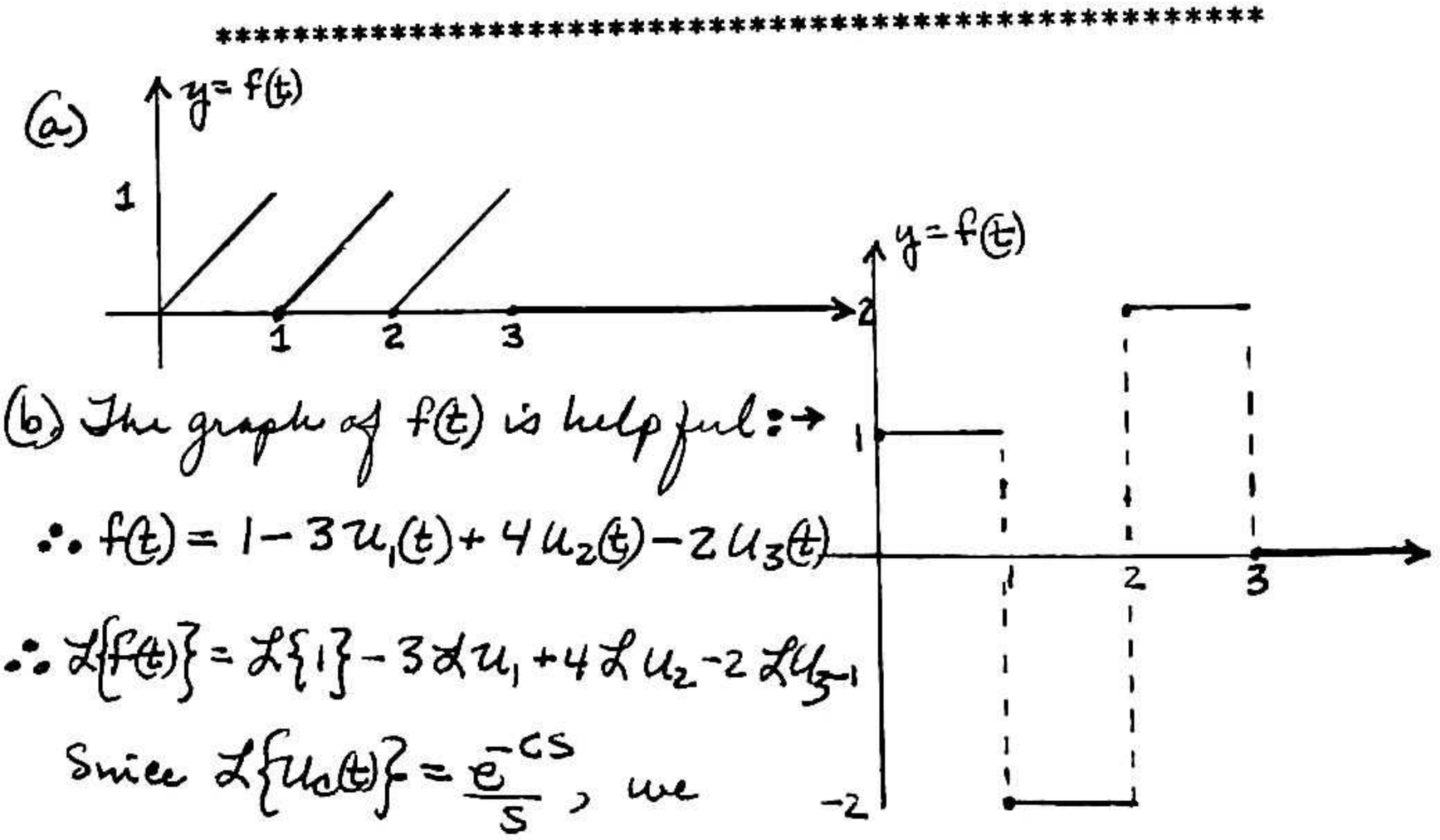
4-37

(a) Sketch a graph of $f(t) = t - u_1(t) - u_2(t) + (2 - t)u_3(t)$ for $t \geq 0$.

$$\text{Recall } U_c(t) = \begin{cases} 0 & t < c \\ 1 & t \geq c \end{cases}$$

$$\text{(b) Find } L\{f(t)\} \text{ if } f(t) = \begin{cases} 1 & 0 \leq t < 1 \\ -2 & 1 \leq t < 2 \\ +2 & 2 \leq t < 3 \\ 0 & 3 \leq t \end{cases}$$

****



(b) The graph of $f(t)$ is helpful :→

$\therefore f(t) = 1 - 3u_1(t) + 4u_2(t) - 2u_3(t)$

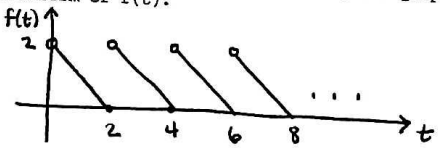
$\therefore \mathcal{L}\{f(t)\} = \mathcal{L}\{1\} - 3\mathcal{L}u_1 + 4\mathcal{L}u_2 - 2\mathcal{L}u_3$

Since $\mathcal{L}\{u_c(t)\} = \frac{e^{-cs}}{s}$, we have $\mathcal{L}\{f(t)\} = \frac{1}{s} - 3\cdot\frac{e^{-s}}{s} + 4\frac{e^{-2s}}{s} - 2\frac{e^{-3s}}{s}$

$\therefore \mathcal{L}\{f(t)\} = \frac{1}{s}\left[1 - 3e^{-s} + 4e^{-2s} - 2e^{-3s}\right]$

4-36

The periodic function f(t) is represented by the graph below. Find the Laplace transform of f(t).



****

The period of this function is 2. $f(t)$ for $t \in [0,2]$ is defined by $f(t) = 2-t$ (the equation of the line through $(0,2)$ & $(2,0)$ has slope $\frac{2-0}{0-2} = -1$ & has intercept $(0,2)$ so is $y = -t+2$.)

$$\mathcal{L}\{f(t)\} = \frac{1}{1-e^{-2s}} \int_0^2 (2-t)\, e^{-st}\, dt$$

$u = 2-t \quad dv = e^{-st}dt$

$du = -dt \quad v = -\frac{1}{s}e^{-st}$

$$= \frac{1}{1-e^{-2s}} \left[(2-t)\left(-\frac{1}{s}e^{-st}\right) - \frac{1}{s}\int e^{-st}\, dt \right]$$

$$= \frac{1}{1-e^{-2s}} \left[(2-t)\left(-\frac{1}{s}e^{-st}\right) + \frac{1}{s^2}e^{-st} \right]_0^2$$

$$= \frac{1}{1-e^{-2s}} \left[0\left(-\frac{1}{s}e^{-2s}\right) + \frac{1}{s^2}e^{-2s} - \left(2\left(-\frac{1}{s}e^{0}\right) + \frac{1}{s^2}e^{0}\right) \right]$$

$$= \frac{1}{1-e^{-2s}} \left[\frac{1}{s^2}e^{-2s} + \frac{2}{s} - \frac{1}{s^2} \right]$$

4-39

Find the Laplace transform of $r(x) = \begin{cases} x & 0 < x < 1 \\ 1 & x > 1 \end{cases}$ two ways:

(a) directly from the definition of the Laplace transform

(b) using Heaviside functions.

****

(a) $\mathcal{L}(r(x)) = \int_0^\infty r(x)\, e^{-px}\, dx =$

$\int_0^1 x\, e^{-px}\, dx + \int_1^\infty e^{-px}\, dx =$

$\left(\frac{-x}{p} e^{-px} - \frac{1}{p^2} e^{-px} \right|_0^1 + \left(\frac{-1}{p} e^{-px} \right|_1^\infty =$

$\left(\frac{-e^{-p}}{p} - \frac{e^{-p}}{p^2} \right) - \left(\frac{-1}{p^2} \right) + (0) - \left(\frac{-e^{-p}}{p} \right)$

$= \frac{1}{p^2} - \frac{e^{-p}}{p^2} = \boxed{\frac{1 - e^{-p}}{p^2}}$

(b) $r(x) =$ (thing 1) $+ H(x-1)$ (thing 2) since there is a change at 1. For $x < 1$, $H(x-1) = 0$ and we get $r(x) =$ thing 1. We want x, so thing 1 $= x$. For $x > 1$, $H(x-1) = 1$ and we get $x +$ thing 2. We want 1, so thing 2 $= 1 - x$. Thus:

4-38

The **unit gate function** $G_a(t)$ is defined by

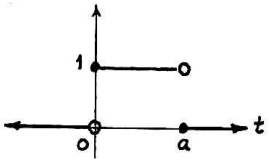
$$G_a(t) = \begin{cases} 0 & t < 0 \\ 1 & 0 \le t < a \\ 0 & a \le t \end{cases}$$

(a) Write $G_a(t-b)$ in terms of the unit step function ($\mathcal{U}$).

(b) Determine $\mathcal{L}\{\, G_a(t-b)\,\}$

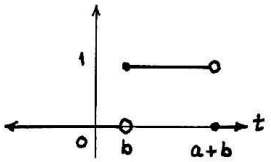
****

a) Consider the graph of $G_a(t)$:



And note that the graph of $G_a(t-b)$ can be obtained by translating the graph of $G_a(t)$ b units to the right.

Thus, the graph of $G_a(t-b)$:



From the graph: $\underline{\underline{G_a(t-b) = \mathcal{U}(t-b) - \mathcal{U}[t-(a+b)]}}$

b) Now, $\mathcal{L}\{G_a(t-b)\} = \mathcal{L}\{\mathcal{U}(t-b)\} - \mathcal{L}\{\mathcal{U}[t-(a+b)]\}$

$$= \frac{1}{s}e^{-bs} - \frac{1}{s}e^{-(a+b)s}$$

$$= \underline{\underline{\frac{1}{s}\left(e^{-bs} - e^{-(a+b)s}\right)}}$$

4-41

Compute the Laplace Transform of the function f(t) whose graph is

****

Method 1: The Heaviside function H(t-a) has graph

So we can write f(t) as

$$f(t) = H(t) - H(t-a) + H(t-2a) - H(t-3a) + \cdots$$

We use the fact that $\mathcal{L}\{H(t-a)\} = \frac{e^{-as}}{s}$

to get

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{H(t)\} - \mathcal{L}\{H(t-a)\} + \mathcal{L}\{H(t-2a)\} - \mathcal{L}\{H(t-3a)\} + \cdots$$

$$= \frac{1}{s} - \frac{e^{-as}}{s} + \frac{e^{-2as}}{s} - \frac{e^{-3as}}{s} + \cdots$$

$$= \frac{1}{s}\left[1 - e^{-as} + e^{-2as} - e^{-3as} + \cdots\right]$$

$$= \frac{1}{s}\left[1 - e^{-as} + (e^{-as})^2 - (e^{-as})^3 + \cdots\right]$$

This is the geometric series $\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots$ with $x = -e^{-as}$. Therefore

$r(x) = x + H(x-1)(1-x)$.

$\mathcal{L}(r(x)) = \mathcal{L}(x) + \mathcal{L}(H(x-1)(1-x))$.

$\mathcal{L}(H(x-a)g(x)) = e^{-ap}\mathcal{L}(g(x+a))$, so

with $a=1$ $\mathcal{L}(H(x-1)(1-x)) =$

$e^{-p}\mathcal{L}(1-(x+1)) = e^{-p}\mathcal{L}(-x) = e^{-p}\cdot\frac{-1}{p^2}$

Thus $\mathcal{L}(r(x)) = \frac{1}{p^2} + e^{-p}\cdot\frac{-1}{p^2} = \boxed{\frac{1-e^{-p}}{p^2}}$

4-40

$$f(t) = \begin{cases} \frac{1}{2}t & , \ 0 \le t < 2 \\ 2 & , \ 2 \le t < 3 \\ 0 & , \ t \ge 3 \end{cases}$$

a) Represent f(t) in terms of unit step functions.

b) Find $\mathcal{L}\{f(t)\}$.

****

a) $f(t) = \frac{1}{2}t - \frac{1}{2}t\,u(t-2) + 2u(t-2) - 2u(t-3)$

b) $\mathcal{L}\{f(t)\} = \mathcal{L}\{\frac{1}{2}t - \frac{1}{2}t\,u(t-2) + 2u(t-2) - 2u(t-3)\}$

$= \mathcal{L}\{\frac{1}{2}t - \frac{1}{2}(t-2)u(t-2) + u(t-2) - 2u(t-3)\}$

$= \frac{1}{2s^2} - \frac{1}{2s^2}e^{-2s} + \frac{1}{s}e^{-2s} - \frac{2}{s}e^{-3s}$

4-42

$$\frac{dy}{dt} = t\ ,\ 0 \leq t < 1$$
$$= 0,\qquad t \geq 1 \qquad\qquad y(0) = 0.$$

$y(t) =$ (A) $\frac{1}{2}t^2 - \frac{1}{2}(t-1)^2H(t-1) - (t-1)H(t-1)$

(B) $\frac{1}{2}(t-1)^2 - \frac{1}{2}(t-1)^2H(t-1) - (t-1)H(t-1)$

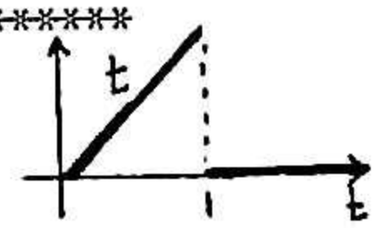
(C) $\frac{1}{2}t^2 - \frac{1}{2}t^2H(t-1) - tH(t-1)$

(D) $\frac{1}{2}t^2 - \frac{1}{2}t^2H(t-1) + tH(t-1)$

(E) $\frac{1}{2}t^2 - \frac{1}{2}(t-1)^2H(t-1)$.

****

The function $g(t) = t\ ,\ 0 \leq t < 1$ is the "saw-tooth"
$= 0\ ,\ t \geq 1$



It consists of the line t from which, after $t=1$, t is subtracted so that $g(t)$ will be zero. We subtract $t\,H(t-1)$ where $H(t-1)$ is the Heaviside function that is zero until time $t=1$ and is 1 after $t=1$.

we have $$g(t) = t - t\,H(t-1).$$

We use the formula

$$(\bigstar)\quad f(t-1)\,H(t-1) \longrightarrow F(s)\,e^{-s} \quad \left(\text{where } f(t) \to F(s)\right).$$

Therefore, to transform $t\,H(t-1)$, we need a function of $t-1$ next to $H(t-1)$ and we obtain it by writing

$$t\,H(t-1) = (t-1+1)\,H(t-1) = (t-1)\,H(t-1) + H(t-1).$$

To transform $(t-1)\,H(t-1)$, note that $t-1 \equiv f(t-1)$. Thus,

$f(t) = t \to \frac{1}{s^2} = F(s)$. Hence, applying Formula ★

$$(t-1)\,H(t-1) \longrightarrow \frac{1}{s^2}\,e^{-s}.$$

To transform $H(t-1)$, note that $1 \equiv f(t-1)$. Denote $u \equiv t-1$

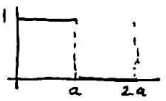
$$\mathcal{L}\{f(t)\} = \frac{1}{s} \cdot \frac{1}{1-(-e^{-as})} = \frac{1}{s} \cdot \frac{1}{1+e^{-as}}.$$

Method 2: We recognize that $f(t)$ is a periodic function with period $\omega = 2a$. The Laplace Transform of a periodic function with period ω is

$$\mathcal{L}\{f(t)\} = \frac{\int_0^\omega e^{-st} f(t)\,dt}{1-e^{-\omega s}}.$$

$f(t)$ over one period is

$$f(t) = \begin{cases} 1, & 0 < t < a \\ 0, & a < t < 2a \end{cases}$$



We compute $\int_0^\omega e^{-st} f(t)\,dt$

$$\int_0^{2a} e^{-st} f(t)\,dt = \int_0^a e^{-st} \cdot 1\,dt + \int_a^{2a} e^{-st} \cdot 0\,dt$$

$$= \int_0^a e^{-st}\,dt = -\frac{1}{s} e^{-st}\Big|_0^a$$

$$= \left[-\frac{1}{s}e^{-as}\right] - \left[-\frac{1}{s}\cdot 1\right] = \frac{1-e^{-as}}{s}$$

So

$$\mathcal{L}\{f(t)\} = \frac{\frac{1-e^{-as}}{s}}{1-e^{-2as}} = \frac{1}{s} \cdot \frac{1-e^{-as}}{1-e^{-2as}}$$

$$= \frac{1}{s} \cdot \frac{1-e^{-as}}{1-(e^{-as})^2} = \frac{1}{s} \cdot \frac{1-e^{-as}}{(1+e^{-as})(1-e^{-as})}$$

$$= \frac{1}{s} \cdot \frac{1}{1+e^{-as}}.$$

4-43

If $L\{\sin kt\} = \frac{1}{s^2 + k^2}\left[e^{-st}\,[-s \sin kt - k \cos kt]\right]_0^{\infty}$

Find $L\{|\sin 2t|\}$

****

Because $|\sin 2t|$ is periodic with period $= \frac{\pi}{2}$,

$$L\{|\sin 2t|\} = \frac{\int_0^{\frac{\pi}{2}} e^{-st} \sin 2t\, dt}{1 - e^{-\frac{\pi s}{2}}}$$

$$= \frac{1}{1 - e^{-\frac{\pi s}{2}}}\left[\frac{e^{-st}(-s \sin 2t - 2\cos 2t)}{s^2+4}\right]_0^{\frac{\pi}{2}}$$

$$= \frac{2(1 + e^{-\frac{s\pi}{2}})}{(1 - e^{-\frac{s\pi}{2}})(s^2+4)}$$

4-44

Find the inverse Laplace transform of the following:

a) $\mathcal{L}^{-1}\left\{\frac{1}{(s-3)^2}\right\}$

b) $\mathcal{L}^{-1}\left\{\frac{1}{(s-3)^2} e^{-5s}\right\}$

****

a) $\mathcal{L}^{-1}\left\{\frac{1}{(s-3)^2}\right\} = \underline{\underline{t e^{3t}}}$

b) $\mathcal{L}^{-1}\left\{\frac{1}{(s-3)^2} e^{-5s}\right\} = \underline{\underline{(t-5) e^{3(t-5)} U(t-5)}}$

Then $1 = f(u)$. Say $1 = f(17)$. Clearly $f(t)$ is identically 1 (for any t).

We have $f(t) = 1 \longrightarrow \frac{1}{s} = F(s)$. Hence, (by ★), $H(t-1) \longrightarrow \frac{1}{s} e^{-s}$.

Now we transform the equation

$$\frac{dy}{dt} = t - t\,H(t-1) = t - \left[(t-1)H(t-1) + H(t-1)\right]$$

$$= t - (t-1)H(t-1) - H(t-1).$$

$$\frac{dy}{dt} \longrightarrow sY - \underbrace{y(0)}_{=0} \qquad t \longrightarrow \frac{1}{s^2} \qquad (t-1)H(t-1) \longrightarrow \frac{1}{s^2}e^{-s} \qquad H(t-1) \longrightarrow \frac{1}{s}e^{-s}.$$

We obtain the equation:

$$sY(s) = \frac{1}{s^2} - \frac{1}{s^2}e^{-s} - \frac{1}{s}e^{-s}$$

Solving for $Y(s)$:

$$Y(s) = \frac{1}{s^3} - \frac{1}{s^3}e^{-s} - \frac{1}{s^2}e^{-s}.$$

and we use formula (★) to transform back to t.

$$\underbrace{\frac{1}{s^3}}_{\frac{1}{2}t^2 = f(t)}\, e^{-s} \longrightarrow \underbrace{\frac{1}{2}(t-1)^2}_{f(t-1)}\, H(t-1).$$

; We need $f(t-1)$

$$\underbrace{\frac{1}{s^2}}_{t = f(t)}\, e^{-s} \longrightarrow \underbrace{(t-1)}_{f(t-1)}\, H(t-1).$$

; We need $f(t-1)$

In conclusion,

$$y(t) = \frac{1}{2}t^2 - \frac{1}{2}(t-1)^2 H(t-1) - (t-1)H(t-1)$$

The correct answer is (A). Incorrect use of the key formula (★) may lead to any of the other wrong answers.

4-46

$F(s) = \arctan \frac{1}{2}s$ $\quad f(t) =$ (A) $(1/t)\sin 2t$

(B) $-(2/t)\sin 2t$

(C) $-(1/t)\sin 2t$

(D) $-t\sin\frac{1}{2}t$

(E) $-(2/t)\sin\frac{1}{2}t$

****

We attempt to transform the derivative of $F(s)$ by the formula

$$(*) \quad t f(t) \longrightarrow -\frac{d}{ds} F(s) \quad (\text{where } f(t) \to F(s)).$$

To find the derivative of

$$y = \arctan x,$$

note that

$$\tan y = x.$$

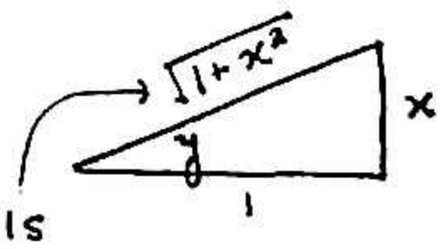
Differentiating both sides with respect to y we obtain

$$\frac{1}{\cos^2 y} = \frac{dx}{dy}$$

Thus,

$$\frac{dy}{dx} = \cos^2 y.$$

We need $y'(x)$ in terms of x. To that end we draw the triangle in which $\tan y = x$.



In this triangle the hypotenuse is and $\cos y = \frac{1}{\sqrt{1+x^2}}$. Hence, $\cos^2 y = \frac{1}{1+x^2}$ and

$$\frac{d}{dx}(\arctan x) = \frac{1}{1+x^2}$$

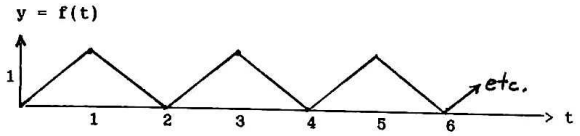
Therefore, by the chain rule for differentiation,

DIFFERENTIATION AND INTEGRATION

4-45

For the function $y = f(t)$ graphed below, find Lf.

(Hint: Recall $Lf' = sLf - f(0)$).



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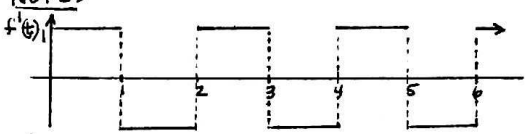
Note:



Hence, if

$$u_c(t) = \begin{cases} 0 & t < c \\ 1 & t \geq c \end{cases}$$

$$\mathcal{L}u_c = \frac{e^{-cs}}{s}$$

$$\therefore f'(t) = 1 - 2u_1 + 2u_2 - 2u_3 + 2u_4 - \cdots$$

$$\mathcal{L}f'(t) = \mathcal{L}\{1\} - 2\mathcal{L}u_1 + 2\mathcal{L}u_2 - 2\mathcal{L}u_3 + 2\mathcal{L}u_4 - \cdots$$

$$= \frac{1}{s} - 2\frac{e^{-s}}{s} + 2\frac{e^{-2s}}{s} - 2\frac{e^{-3s}}{s} + 2\frac{e^{-4s}}{s} - \cdots$$

$$= \frac{1}{s} - \frac{2}{s}e^{-s}\underbrace{\{1 - e^{-s} + e^{-2s} - e^{-3s} + \cdots\}}_{\text{geometric series with } a=1,\ \text{common ratio } r = -e^{-s}} \quad (*)$$

and since $a + ar + ar^2 + \cdots = \frac{a}{1-r}$, $(*)$ becomes:

$$\mathcal{L}f'(t) = \frac{1}{s} - \frac{2}{s}e^{-s}\left\{\frac{1}{1+e^{-s}}\right\} = s\mathcal{L}f - \overset{0}{\overset{\shortparallel}{f(0)}} \quad \text{(from hint)}$$

$$\therefore \mathcal{L}f = \frac{1}{s}\mathcal{L}f' = \frac{1}{s}\left[\frac{1}{s} - \frac{2}{s}e^{-s}\left\{\frac{1}{1+e^{-s}}\right\}\right] = \frac{1}{s^2}\left[1 - \frac{2e^{-s}}{1+e^{-s}}\right]$$

$$= \frac{1}{s^2}\left[\frac{1+e^{-s}-2e^{-s}}{1+e^{-s}}\right] = \frac{1}{s^2}\left[\frac{1-e^{-s}}{1+e^{-s}}\right] \quad \therefore \mathcal{L}f = \frac{1}{s^2}\left[\frac{1-e^{-s}}{1+e^{-s}}\right]$$

Then $f''(s) = D_s[-3(s-2)^{-2}] = 6(s-2)^{-3}$

$= \frac{6}{(s-2)^3}$, $s > 2$

$\Rightarrow \mathcal{L}\{3t^2e^{2t}\} = \frac{6}{(s-2)^3}$, $s > 2$.

4-48

Find the Laplace transform of $\frac{\sin(bt)}{t}$, $t \neq 0$, $b > 0$.

****

Let $f(t) = \sin bt$ and $F(s) = \mathcal{L}\{f\}(s) = \frac{b}{s^2+b^2}$.

Then f is piecewise continuous and of exponential order on $[0, \infty)$.

$\lim_{t \to 0+} \frac{f(t)}{t} = b$ is finite.

$\therefore \quad \mathcal{L}\left\{\frac{f(t)}{t}\right\}(s) = \int_s^\infty F(u)\,du$

$= \int_s^\infty \frac{b}{u^2+b^2}\,du$

$= \tan^{-1}\frac{u}{b}\Big|_s^\infty$

$= \frac{\pi}{2} - \tan^{-1}\frac{s}{b}$, $s > 0$.

$$\frac{d}{ds}\left(\arctan\frac{s}{2}\right) = \frac{1}{1+\left(\frac{s}{2}\right)^2}\cdot\underbrace{\frac{d\left(\frac{s}{2}\right)}{ds}}_{=\frac{1}{2}} \quad \leftarrow \text{need that by the chain rule}$$

$$= \frac{2}{2^2+s^2}$$

We now apply (*)

$$\underbrace{-\frac{d}{ds}\left(\arctan\frac{s}{2}\right)}_{\downarrow\; t\,f(t)} = \frac{-2}{s^2+2^2} \longrightarrow -\sin 2t$$

Hence,

$$t\,f(t) = -\sin 2t$$

$$f(t) = \frac{-\sin 2t}{t}$$

The correct answer is (C).

4-47

Find the Laplace transform of $3t^2e^{2t}$

****

Solution: $L\{3t^2e^{2t}\} = L\{(-t)^2 3e^{2t}\}$

from $L\{(-t)^n F(t)\} = f^{(n)}(s)$ where $f(s) = L\{F(t)\}$

and from $L\{3e^{2t}\} = 3L\{e^{2t}\} = 3\frac{1}{s-2} = f(s)$, $(s>2)$,

4-50

Let $f(t) = \sin^2 t = (\sin t)^2$ and $g(t) = \cos^2 t = (\cos t)^2$.

a. Find $L\{f(t)\}$, the Laplace Transform of f.

b. Find $L\{g(t)\}$, the Laplace Transform of g.

[Hint: Consider the derivative of f, f'.]

****

a. $f'(t) = 2\sin t\cos t = \sin 2t$

$\mathcal{L}\{f'(t)\} = \mathcal{L}\{\sin 2t\} = \frac{2}{s^2+4}$; also,

$\mathcal{L}\{f'(t)\} = s\mathcal{L}\{f(t)\} - f(0)$, but $f(0) = 0$.

Thus $s\mathcal{L}\{f(t)\} = \frac{2}{s^2+4}$, and

$\mathcal{L}\{f(t)\} = \mathcal{L}\{\sin^2 t\} = \frac{2}{s(s^2+4)}$.

b. $g(t) = 1 - f(t)$, thus $\mathcal{L}\{g(t)\} = \mathcal{L}\{1\} - \mathcal{L}\{f(t)\}$

$\mathcal{L}\{g(t)\} = \frac{1}{s} - \frac{2}{s(s^2+4)} = \frac{s^2+4-2}{s(s^2+4)}$

$\mathcal{L}\{g(t)\} = \mathcal{L}\{\cos^2 t\} = \frac{s^2+2}{s(s^2+4)}$.

(Comment: Using partial fractions, the answers may also be given as:

$\mathcal{L}\{\sin^2 t\} = \frac{1}{2}\left(\frac{1}{s} - \frac{s}{s^2+4}\right)$ and

$\mathcal{L}\{\cos^2 t\} = \frac{1}{2}\left(\frac{1}{s} + \frac{s}{s^2+4}\right)$.)

4-49

$\frac{dy}{dt} + \int_0^t y(u)du = \sin t, \quad y(0) = 0.$

$y(t) =$ (A) $-\frac{1}{2}t\sin t$

(B) $\frac{1}{2}t\sin t$

(C) $-(1/2t)\sin t$

(D) $(1/2t)\sin t$

(E) $\frac{1}{2}t\cos t$

****

Since $\frac{dy}{dt} \rightarrow sY - \underbrace{y(0)}_{0}$

$\int_0^t y(u)du \rightarrow \frac{1}{s}Y(s)$,

The equation transforms to

$$sY + \frac{1}{s}Y = \frac{1}{s^2+1}$$

$$Y\underbrace{\left(s + \frac{1}{s}\right)}_{\frac{s^2+1}{s}} = \frac{1}{s^2+1}$$

$$Y = \frac{s}{(s^2+1)^2} = \frac{1}{2}\left(-\frac{d}{ds}\frac{1}{s^2+1}\right)$$

Now, since $\frac{1}{s^2+1} \rightarrow \sin t$,

then $-\frac{d}{ds}\left(\frac{1}{s^2+1}\right) \rightarrow t\sin t$.

Therefore, the equation transforms into

$$y(t) = \frac{1}{2}t\sin t$$

This is the answer (B).

CONVOLUTION

4-52

(a) Find $L\left\{\int_0^t \sin(t-\tau)\, e^{4\tau}\, d\tau\right\}$

(b) If $y(t) + \int_0^t k(t-s)y(s)ds = g(t)$, solve for $y(t)$.

Answer may be left in the form $y = L^{-1}\{\ \}$. Assume $Lk = K(s)$, $Lg = G(s)$.

****

(a) $\mathcal{L}\{\underbrace{\int_0^t \sin(t-\tau)e^{4\tau}d\tau}_{f*g}\} \overset{\cdot\cdot}{=} F(s)\,G(s) = \frac{1}{s^2+1}\cdot\frac{1}{(s-4)}$

(this is the convolution of f and g

where $f = \sin(t)$, $g = e^{4t}$

$\updownarrow$ $\updownarrow$

$F(s) = \frac{1}{s^2+1}$ $\quad$ $G(s) = \frac{1}{s-4}$

(b.) $y(t) + \int_0^t k(t-s)\,y(s)ds = g(t)$

$\mathcal{L}$ (

$\mathcal{L}y + \mathcal{L}\{\int_0^t k(t-s)\,y(s)ds\} = \mathcal{L}g$

$\mathcal{L}y + \mathcal{L}\{k*y\} = \mathcal{L}g$

$\mathcal{L}y + \mathcal{L}\{k\}\cdot\mathcal{L}\{y\} = \mathcal{L}g$

$[1 + \mathcal{L}\{k\}]\,\mathcal{L}\{y\} = \mathcal{L}g \Rightarrow \mathcal{L}\{y\} = \frac{\mathcal{L}g}{1+\mathcal{L}\{k\}}$

$\therefore \mathcal{L}\{y\} = \frac{G(s)}{1+K(s)}$

$\therefore \underline{y(t) = \mathcal{L}^{-1}\left\{\frac{G(s)}{1+K(s)}\right\}}$

4-51

$tf(t) = \cos t/t$ $\quad F(s) =$ (A) $\frac{1}{2}s\ln(s^2+1) - \ln(s^2+1)$.

(B) $\frac{1}{2}s\ln(s^2+1)$

(C) $\frac{1}{2}s\ln(s^2+1) - s + \arctan s$.

(D) $-\sin s - s^2 + \ln(s^2+1)$

(E) $s\cos s - s + \ln(s^2+1)$

****

$$t^2 f(t) = \cos t$$

Note that

$$t^2 f(t) = t \underbrace{t f(t)}_{\downarrow -\frac{d}{ds}F(s)} \rightarrow -\frac{d}{ds}\left(-\frac{d}{ds}F(s)\right) = F''(s).$$

While

$$\cos t \rightarrow \frac{s}{s^2+1}$$

Hence,

$$F''(s) = \frac{s}{s^2+1}$$

Integrating

$$F'(s) = \int \frac{s}{s^2+1}\,ds \underset{\uparrow}{=} \frac{1}{2}\int \frac{du}{u} = \frac{1}{2}\ln u = \frac{1}{2}\ln(s^2+1).$$

$u = s^2+1$

$\frac{du}{ds} = 2s \Rightarrow \frac{1}{2}du = s\,ds$

To integrate the log function we use integration by parts

$$F(s) = \frac{1}{2}\int \ln(s^2+1)\,ds = \frac{1}{2}s\ln(s^2+1) - \frac{1}{2}\int s\,\frac{2s}{s^2+1}\,ds$$

$$= \frac{1}{2}s\ln(s^2+1) - \int \underbrace{\frac{s^2}{s^2+1}}_{\text{(1)}}\,ds$$

$$\frac{s^2+1-1}{s^2+1} = \underbrace{\frac{s^2+1}{s^2+1}}_{\text{(1)}} - \frac{1}{s^2+1}$$

$$= \frac{1}{2}s\ln(s^2+1) - \underbrace{\int ds}_{s} + \underbrace{\int \frac{1}{s^2+1}\,ds}_{\arctan s.} \quad ; \text{ Answer (C).}$$

4-55

Find the LaPlace Transform of $\int_0^t (t-z)\,(\sin 5z)\,dz$

****

This is the convolution of t and $\sin 5t$.

$$\mathcal{L}\{t * \sin 5t\} = \frac{1}{s^2} \cdot \frac{5}{s^2+25} = \frac{5}{s^2(s^2+25)}$$

4-56

Find the following Laplace transforms:

a) $\mathcal{L}\{t^4\}$

b) $\mathcal{L}\{t^4 e^{5t}\}$

c) $\mathcal{L}\left\{\int_0^t 4e^{5(t-\tau)}\,d\tau\right\}$

****

a) $\mathcal{L}\{t^4\} = \frac{4!}{s^5} = \underline{\underline{\frac{24}{s^5}}}$

b) $\mathcal{L}\{t^4 e^{5t}\} = \underline{\underline{\frac{4!}{(s-5)^5}}}$ using the 1st shifting theorem.

c) $\mathcal{L}\left\{\int_0^t \tau^4 e^{5(t-\tau)}\,d\tau\right\} = \mathcal{L}\{t^4 * e^{5t}\}$

$= \mathcal{L}\{t^4\}\,\mathcal{L}\{e^{5t}\} = \frac{4!}{s^5} \cdot \frac{1}{s-5} = \underline{\underline{\frac{24}{s^5(s-5)}}}$

4-53

Solve for y:

$$y' = 2\int_0^x e^{t-x}\, y(t)dt,\ y(0) = 3$$

****

The integral is a convolution;

$y'(x) = 2e^{-x} * y(x)$.

Laplace transforms give:

$$pY(p) - 3 = 2 \cdot \frac{1}{p+1} \cdot Y(p) \Rightarrow$$

$$\left(p - \frac{2}{p+1}\right) Y = 3, \quad \frac{p^2+p-2}{p+1} Y = 3 \Rightarrow$$

$$Y = \frac{3p+3}{p^2+p-2} = \frac{3p+3}{(p+2)(p-1)} = \frac{1}{p+2} + \frac{2}{p-1}$$

Thus $\boxed{y(x) = e^{-2x} + 2e^x}$

4-54

If $f(t) = e^t$, and $g(t) = \sin t$, find the function $y(t)$ for which $f*y = g$.

****

Using capital letters for Laplace transforms,

$F(s) = \frac{1}{s-1}$ and $G(s) = \frac{1}{s^2+1}$, so $\frac{Y(s)}{s-1} = \frac{1}{s^2+1}$.

Hence $Y(s) = \frac{s-1}{s^2+1} = \frac{s}{s^2+1} - \frac{1}{s^2+1}$. $\underline{y(t) = \cos t - \sin t.}$

4-58

Find the inverse Laplace transform of $H(s) = (16s+8)/(s^4+4s^2)$, $s>0$.

****

$$H(s) = \frac{16s+8}{s^2(s^2+4)}$$

$$= \frac{1}{s^2}\cdot\left[16\cdot\frac{s}{s^2+2^2} + 4\cdot\frac{2}{s^2+2^2}\right]$$

$$= F(s)\cdot G(s) \quad , \text{ where } F(s) = \frac{1}{s^2}.$$

$$f(t) = \mathcal{L}^{-1}\{F\}(t) = t$$

$$g(t) = \mathcal{L}^{-1}\{G\}(t) = 16\cos 2t + 4\sin 2t$$

$$h(t) = \mathcal{L}^{-1}\{H\}(t)$$

$$= \mathcal{L}^{-1}\{FG\}(t)$$

$$= (f*g)(t)$$

$$= \int_0^t f(t-u)g(u)\,du$$

$$= \int_0^t (t-u)\left[16\cos 2u + 4\sin 2u\right]du$$

$$= (t-u)\left[8\sin 2u - 2\cos 2u\right]\Big|_{u=0}^{t} - \int_0^t \left[8\sin 2u - 2\cos 2u\right](-1)\,du$$

$$= 2t + \left[-4\cos 2u - \sin 2u\right]\Big|_{u=0}^{t}$$

$$= 2t - 4\cos 2t - \sin 2t + 4.$$

4-57

Find $L^{-1}\left[\frac{1}{(s-2)(s+3)}\right]$ by the method of convolutions. Verify by taking directly the inverse transforms of the partial fractions.

****

$$L\{f(t) * g(t)\} = L\{f(t)\} \times L\{g(t)\}$$
$$= F(s) \times G(s)$$

that is

$$L^{-1}\{F(s) \times G(s)\} = f(t) * g(t)$$

Therefore,

$$= L^{-1}[F(s)] * L^{-1}[G(s)]$$
$$L^{-1}\left\{\frac{1}{s-2} \times \frac{1}{s+3}\right\} = e^{2t} * e^{-3t}$$

Since

$$L\{e^{2t}\} = \frac{1}{s-2}, \quad e^{2t} = L^{-1}\left[\frac{1}{s-2}\right]$$

$$L\{e^{-3t}\} = \frac{1}{s+3}, \quad e^{-3t} = L^{-1}\left[\frac{1}{s+3}\right]$$

But $e^{2t} * e^{-3t} = \int_0^t e^{2z}\, e^{-3(t-z)}\, dz$

$$= e^{-3t} \int_0^t e^{5z}\, dz = e^{-3t}\left[\frac{e^{5z}}{5}\right]_0^t$$
$$= \frac{e^{-3t}}{5}(e^{5t} - 1)$$
$$= \frac{1}{5}(e^{2t} - e^{-3t})$$

that is $L^{-1}\left\{\frac{1}{(s-2)(s+3)}\right\} = \frac{1}{5}(e^{2t} - e^{-3t})$

Check:

$$L^{-1}\left\{\frac{1}{(s-2)(s+3)}\right\} = L^{-1}\left\{\frac{1/5}{s-2} + \frac{-1/5}{s+3}\right\}$$
$$= \frac{1}{5}L^{-1}\left\{\frac{1}{s-2}\right\} - \frac{1}{5}L^{-1}\left\{\frac{1}{s+3}\right\}$$
$$= \frac{1}{5}e^{2t} - \frac{1}{5}e^{-3t}$$
$$= \frac{1}{5}(e^{2t} - e^{-3t})$$

4-60

Find the solution y(t) of the following integral equation by Laplace Transform

$$y(t) + \int_0^t (t-u)\, y(u)\, du = 1 + t\ .$$

****

Let Y(s) denote the Laplace Transform of y(t). Recall the convolution theorem where

$$\mathcal{L}\left\{\int_0^t (t-u)\, y(u)\, du\right\} = \mathcal{L}\{t\}\, Y(s) = \frac{Y(s)}{s^2}$$

Taking Laplace Transform on both sides of the integral equation, we have

$$Y(s) + \frac{Y(s)}{s^2} = \mathcal{L}\{1+t\} = \frac{1}{s} + \frac{1}{s^2} = \frac{s+1}{s^2}\ .$$

Solving for Y(s)

$$Y(s) = \frac{s+1}{s^2+1} = \frac{s}{s^2+1} + \frac{1}{s^2+1}\ .$$

Finally we take the inverse Laplace Transform on both sides to get y(t).

$$y(t) = \mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\}$$

$$= \cos t + \sin t$$

4-59

If the Laplace Transform of $f(t)$ is $\frac{s}{(s^2+9)^2}$, find $f(t)$.

****

IF $L\{g(t)\} = G(s)$ AND $L\{h(t)\} = H(s)$, THEN $L^{-1}\{G(s)\cdot H(s)\}$ IS GIVEN BY $g(t) * h(t) = \int_0^t g(\lambda)\, h(t-\lambda)\, d\lambda$. THIS IS A BASIC "CONVOLUTION" THEOREM.

LET $g(t) = \cos 3t$. THEN $G(s) = \frac{s}{s^2+9}$.

IF $h(t) = \frac{1}{3}\sin 3t$, THEN $H(s) = \frac{1}{s^2+9}$

$$f(t) = L^{-1}\left\{\frac{s}{(s^2+9)^2}\right\} = L^{-1}\left\{\frac{s}{s^2+9}\cdot\frac{1}{s^2+9}\right\} = (\cos 3t) * (\tfrac{1}{3}\sin 3t)$$

$$= \int_0^t (\cos 3\lambda)\left[\tfrac{1}{3}\sin 3(t-\lambda)\right] d\lambda$$

$$= \frac{1}{3}\int_0^t (\cos 3\lambda)(\sin 3t\cos 3\lambda - \cos 3t \sin 3\lambda)\, d\lambda$$

$$= \frac{1}{3}\left[\sin 3t \int_0^t \cos^2 3\lambda\, d\lambda - \cos 3t \int_0^t \sin 3\lambda \cos 3\lambda\, d\lambda\right]$$

$$= \frac{1}{3}\left[\sin 3t\left(\tfrac{1}{2}\lambda + \tfrac{1}{12}\sin 6\lambda\right)\Big|_0^t - \cos 3t\left(\tfrac{1}{6}\sin^2 3\lambda\right)\Big|_0^t\right]$$

$$= \frac{1}{3}\left[\sin 3t\left(\tfrac{t}{2} + \tfrac{1}{12}\sin 6t\right) - \tfrac{1}{6}\sin^2 3t\cos 3t\right]$$

$$= \frac{1}{3}\left[\tfrac{t}{2}\sin 3t + \sin 3t\left(\tfrac{2}{12}\sin 3t\cos 3t\right) - \tfrac{1}{6}\sin^2 3t\cos 3t\right]$$

$$= \tfrac{t}{6}\sin 3t.$$

CHECK: $L\left\{\frac{1}{6}t\sin 3t\right\} = \frac{1}{6}\left(-\frac{d}{ds}\right)\frac{3}{s^2+9} = -\frac{1}{2}\frac{-2s}{(s^2+9)^2}$

$$= \frac{s}{(s^2+9)^2}.$$

4-62

Find $L^{-1}\left[1/(s+2)(s^2+9)\right]$ using a convolution integral.

****

Note that $L^{-1}\left[\frac{1}{s+2}\right] = e^{-2t}$ and that $L^{-1}\left[\frac{1}{s^2+9}\right] = \frac{1}{3}\sin 3t$. Hence the inverse transform of the product $\frac{1}{s+2}\cdot\frac{1}{s^2+9}$ is the convolution of their separate inverses: $L^{-1}\left[\frac{1}{(s+2)(s^2+9)}\right] = e^{-2t} * \frac{1}{3}\sin 3t$

$$= \frac{1}{3}\int_0^t e^{-2(t-\alpha)}\sin 3\alpha\, d\alpha$$

$$= \frac{1}{3}e^{-2t}\int_0^t e^{2\alpha}\sin 3\alpha\, d\alpha$$

$$= \frac{1}{3}e^{-2t}\left[\frac{e^{2\alpha}}{13}(2\sin 3\alpha - 3\cos 3\alpha)\right]_0^t$$

$$= \frac{1}{3}e^{-2t}\left[\frac{e^{2t}}{13}(2\sin 3t - 3\cos 3t) + \frac{3}{13}\right]$$

$$= \frac{2}{39}\sin 3t - \frac{1}{13}\cos 3t + \frac{1}{13}e^{-2t}.$$

4-61

$\int_0^t u(t-u)^3 du =$ (A) $\frac{1}{4}t^5$

(B) $(1/20)t^5$

(C) $(1/5)\ t^5$

(D) $(1/20)t^5 - (1/6)t^3$

(E) $(1/20)t^5 + (1/6)t^3$

****

The long and winding road to the evaluation of this integral is by expanding $(t-u)^3$. Instead, we observe that the integral is the convolution of $f(t) = t$ and $g(t) = t^3$. By the convolution theorem the convolution transforms to the product $F(s)\,G(s)$ that is

$$\begin{array}{ccc} t * t^3 & \longrightarrow & \frac{1}{s^2}\cdot\frac{6}{s^4} = \frac{6}{s^6}. \\ \downarrow \quad \downarrow & & \\ \frac{1}{s^2} \quad \frac{6}{s^4} & & \end{array}$$

Transforming back to t

$$\frac{6}{s^6} \rightarrow \frac{1}{20}t^5$$

Note, $t \rightarrow \frac{1}{s^2}$
$t^2 \rightarrow 2/s^3$
$t^3 \rightarrow 2\cdot3/s^4$
$t^4 \rightarrow 2\cdot3\cdot4/s^5$
$t^5 \rightarrow 2\cdot3\cdot4\cdot5/s^6$.

Thus, $$\int_0^t u(t-u)^3 du = \frac{1}{20}t^5.$$

The correct answer is (B).

4-64

Compute $\mathcal{L}^{-1}\left\{\frac{s}{(s^2+1)^2}\right\}$, the inverse Laplace Transform of $\frac{s}{(s^2+1)^2}$.

****

$\frac{s}{(s^2+1)^2} = \frac{1}{s^2+1} \cdot \frac{s}{s^2+1}$ is the product of two transforms. Hence we can apply the Convolution Theorem –

$$\mathcal{L}\{(f * g)(t)\} = F(s)\,G(s)$$

where $F(s) = \mathcal{L}\{f(t)\}$, $G(s) = \mathcal{L}\{g(t)\}$

and $(f * g)(t) = \int_0^t f(t-u)\,g(u)\,du$.

Here $F(s) = \frac{1}{s^2+1}$ and $G(s) = \frac{s}{s^2+1}$.

Table look-up gives $f(t) = \sin t$ and $g(t) = \cos t$.

$$\sin t * \cos t = \mathcal{L}^{-1}\left\{\frac{1}{s^2+1} \cdot \frac{s}{s^2+1}\right\}.$$

We must compute $\sin t * \cos t$.

$$\sin t * \cos t = \int_0^t \sin(t-u)\cos u\,du$$

Using $\sin(a-b) = \sin a \cos b - \sin b \cos a$

we get

$$\sin t * \cos t = \int_0^t (\sin t \cos u - \sin u \cos t)\cos u\,du$$

4-63

Use Laplace Transforms to solve the equation $F(t)=1+2\int_0^t F(t-B)\cos B\,dB$ used

****

Taking the transform of both sides:

$$f(s) = \frac{1}{s} + 2f(s)\cdot\frac{s}{s^2+1}$$

$$f(s)\left[1 - \frac{2s}{s^2+1}\right] = \frac{1}{s}$$

$$f(s)\,\frac{(s-1)^2}{s^2+1} = \frac{1}{s}$$

$$f(s) = \frac{s^2+1}{s(s-1)^2} = \frac{A}{s} + \frac{B}{s-1} + \frac{C}{(s-1)^2}$$

$$s^2+1 = As^2 - 2As + A + Bs^2 - Bs + Cs$$

$$\left.\begin{aligned} A + B &= 1 \\ -2A - B + C &= 0 \\ A &= 1 \end{aligned}\right\}$$

Solving this system

$A=1,\ B=0,\ C=2$

$$F(t) = L^{-1}\left\{\frac{1}{s} + \frac{2}{(s-1)^2}\right\} = 1 + 2te^t$$

4-65

Use the Method of Convolution to obtain the inverse Laplace transform of $\frac{1}{s^2 - 4}$.

****

The convolution equation is given by

$$F * G = \int_0^t F(u)\,G(t-u)\,du$$

and $\frac{1}{s^2-4} = \frac{1}{s-2} \cdot \frac{1}{s+2}$

Choose F + G arbitrarily as follows

$F = \mathcal{L}^{-1}\left\{\frac{1}{s-2}\right\}$; $G = \mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\}$

$F = e^{2t}$ $\qquad$ $G = e^{-2t}$

$F(u) = e^{2u}$ $\qquad$ $G(t-u) = e^{-2(t-u)}$

Therefore $F * G = \int_0^t e^{2u} e^{-2(t-u)}\,du$

$= \int_0^t e^{2u} \cdot e^{-2t} \cdot e^{2u}\,du$

$= e^{-2t} \int_0^t e^{4u}\,du$

$= e^{-2t}\left[\frac{1}{4} e^{4u}\right]_0^t$

$= e^{-2t}\left(\frac{1}{4} e^{4t} - \frac{1}{4}\right)$

$= \frac{1}{4} e^{2t} - \frac{1}{4} e^{-2t}$

$= \frac{1}{2}\left(\frac{e^{2t} - e^{-2t}}{2}\right)$

$= \underline{\underline{\frac{1}{2} \sinh 2t}}$

$$= \sin t \int_0^t \cos^2 u \, du - \cos t \int_0^t \sin u \cos u \, du$$

We use the following trig identities (or look up the integrals in a table)

$$\cos^2\theta = \tfrac{1}{2}(1+\cos 2\theta), \quad \sin 2\theta = 2\sin\theta\cos\theta$$

$$\sin t * \cos t = \sin t \int_0^t \tfrac{1}{2}(1+\cos 2u)\,du - \cos t \int_0^t \tfrac{1}{2}\sin 2u \, du$$

$$= \tfrac{1}{2}\sin t \int_0^t du + \tfrac{1}{2}\sin t \int_0^t \cos 2u\,du - \tfrac{1}{2}\cos t \int_0^t \sin 2u\,du$$

$$\int du = u, \quad \int \cos 2u\,du = \tfrac{1}{2}\sin 2u, \quad \int \sin 2u\,du = -\tfrac{1}{2}\cos 2u$$

$$\sin t * \cos t = \tfrac{1}{2}\sin t \; u\Big|_0^t + \tfrac{1}{2}\sin t \cdot \tfrac{1}{2}\sin 2u\Big|_0^t - \tfrac{1}{2}\cos t \cdot (-\tfrac{1}{2})\cos 2u\Big|_0^t$$

$$= \tfrac{1}{2}t\sin t + \tfrac{1}{4}\sin t \sin 2t + \tfrac{1}{4}\cos t \cos 2t - \tfrac{1}{4}\cos t$$

Using the above trig identities we get

$$\sin t * \cos t = \tfrac{1}{2}t\sin t + \tfrac{1}{4}\sin t(2\sin t\cos t) + \tfrac{1}{4}\cos t(2\cos^2 t - 1) - \tfrac{1}{4}\cos t$$

$$= \tfrac{1}{2}t\sin t + \tfrac{1}{2}\sin^2 t\cos t + \tfrac{1}{2}\cos t\cos^2 t - \tfrac{1}{2}\cos t$$

$$= \tfrac{1}{2}t\sin t + \tfrac{1}{2}(\sin^2 t + \cos^2 t)\cos t - \tfrac{1}{2}\cos t$$

$$= \tfrac{1}{2}t\sin t + \tfrac{1}{2}\cos t - \tfrac{1}{2}\cos t$$

$$= \tfrac{1}{2}t\sin t.$$

4-67

Solve by Laplace Transform methods. $y''(t)+y(t)=6\sin 2t$ where $y(0)=3$ and $y'(0)=1$

****

Taking transforms of both sides,

$$s^2F(s) - s\,y(0) - y'(0) + F(s) = \frac{12}{s^2+4}$$

$$(s^2+1)F(s) = 3s+1 + \frac{12}{s^2+4}$$

$$F(s) = \frac{3s^3+s^2+12s+16}{(s^2+1)(s^2+4)} = \frac{As+B}{s^2+1} + \frac{Cs+D}{s^2+4}$$

$$3s^3+s^2+12s+16 = As^3+Bs^2+4As+4B+Cs^3+Ds^2+Cs+D$$

$$\left.\begin{aligned} A \quad +C \quad &= 3 \\ B \quad +D \quad &= 1 \\ 4A \quad +C \quad &= 12 \\ 4B \quad +D \quad &= 16 \end{aligned}\right\}$$ Solving This system:

$A=3,\ C=0,\ B=5,\ D=-4$

$$F(s) = 3\left(\frac{s}{s^2+1}\right) + 5\left(\frac{1}{s^2+1}\right) - 2\left(\frac{2}{s^2+4}\right)$$

$$y = 3\cos t + 5\sin t - 2\sin 2t$$

SOLUTION OF DIFFERENTIAL EQUATIONS WITH INITIAL CONDITIONS

4-66

Solve for y(t) : $\frac{dy}{dt} + 4\int_0^t y(\tau)\,d\tau = 1$, $y(0) = 1$

****

First note that $\int_0^t y(\tau)\,d\tau = y(t) * 1$, the convolution of $y(t)$ & 1. Now take the Laplace transform of the equation:

$$sY(s) - y(0) + 4Y(s)\cdot\frac{1}{s} = \frac{1}{s}$$

$$sY(s) - 1 + 4Y(s)\cdot\frac{1}{s} = \frac{1}{s}$$

$$\left(s + \frac{4}{s}\right)Y(s) = \frac{1}{s} + 1 \quad \Rightarrow \quad \frac{s^2+4}{s}Y(s) = \frac{1}{s} + 1$$

Divide both sides by $\frac{s^2+4}{s}$ so

$$Y(s) = \frac{1}{s^2+4} + \frac{s}{s^2+4}$$ & now take the inverse Laplace transform of both sides

$$y(t) = \underline{\underline{\frac{1}{2}\sin 2t + \cos 2t.}}$$

4-69

Solve the initial value problem:

$$y'' + 9y = \begin{cases} 2 & , \quad 0 \leq t < 5 \\ 0 & , \quad 5 \leq t < \infty \end{cases} \qquad y(0) = 0, \; y'(0) = 0$$

****

The given equation may be written as

$$y'' + 9y = 2\left[H_0(t) - H_5(t)\right]$$

Taking L.transforms on both sides of the equation

$$s^2 Y(s) - s\,y(0) - y'(0) + 9\,Y(s) = 2\left[\frac{1}{s} - \frac{e^{-5s}}{s}\right]$$

$$(s^2+9)\,Y(s) = 2\left[\frac{1}{s} - \frac{e^{-5s}}{s}\right]$$

$$Y(s) = 2\left[\frac{1}{s(s^2+9)} - \frac{e^{-5s}}{s(s^2+9)}\right]$$

$$= 2\left[\frac{1}{9}\left(\frac{1}{s} - \frac{s}{s^2+9}\right) - \frac{1}{9}\left(\frac{e^{-5s}}{s} - \frac{s\,e^{-5s}}{s^2+9}\right)\right]$$

Taking inverse transforms

$$L^{-1}[Y(s)] = \frac{2}{9}L^{-1}\left[\frac{1}{s}\right] - \frac{2}{9}L^{-1}\left[\frac{s}{s^2+9}\right] - \frac{2}{9}L^{-1}\left[\frac{e^{-5s}}{s}\right] + \frac{2}{9}L^{-1}\left[\frac{s\,e^{-5s}}{s^2+9}\right]$$

$$= \frac{2}{9}(1) - \frac{2}{9}\cos 3t - \frac{2}{9}H_5(t) + \frac{2}{9}H_5(t)\cos 3(t-5)$$

Therefore,

$$Y(t) = \frac{2}{9}\left[1 - \cos 3t - H_5(t)\left(1 - \cos 3(t-5)\right)\right] \text{ is}$$

the solution of the initial value problem.

4-68

Use Laplace transforms to solve the initial-value problem:

$$\frac{d^2y}{dt^2} - 2\frac{dy}{dt} - 8y = 24 \quad \text{with } y(0) = 0,\ y'(0) = 6$$

****

SETTING $\mathcal{L}(y) = Y(s)$, WE HAVE $\mathcal{L}(y') = sY(s) - y(0) = sY(s) - 0$
AND $\mathcal{L}(y'') = s^2Y(s) - sy(0) - y'(0) = s^2Y(s) - 0\cdot s - 6$.

THUS $\mathcal{L}(y'' - 2y' - 8y) = \mathcal{L}(24)$

$$\Rightarrow \left[s^2Y(s) - 6\right] - 2\left[sY(s)\right] - 8\left[Y(s)\right] = \frac{24}{s}$$

$$(s^2 - 2s - 8)Y(s) = \frac{24}{s} + 6 = \frac{24 + 6s}{s}$$

$$Y(s) = \frac{24 + 6s}{s(s^2 - 2s - 8)} = \frac{24 + 6s}{s(s+2)(s-4)}$$

BY PARTIAL FRACTIONS DECOMPOSITION,

$$\frac{24 + 6s}{s(s+2)(s-4)} = \frac{A}{s} + \frac{B}{s+2} + \frac{C}{s-4}$$

$$24 + 6s = A(s+2)(s-4) + Bs(s-4) + Cs(s+2)$$

$s = 0 \Rightarrow 24 = -8A \qquad A = -3$
$s = -2 \Rightarrow 12 = 12B \qquad B = 1$
$s = 4 \Rightarrow 48 = 24C \qquad C = 2$

$$Y(s) = \frac{-3}{s} + \frac{1}{s+2} + \frac{2}{s-4}$$

SPECIFIC SOLUTION:

$$y(t) = \mathcal{L}^{-1}(Y(s)) = \mathcal{L}\left(\frac{-3}{s} + \frac{1}{s+2} + \frac{2}{s-4}\right)$$

$$= -3 + e^{-2t} + 2e^{4t}$$

4-71

Solve $ty' + y = \cosh t$ with $y(0) = 1$.

****

Using $\mathcal{L}\{t^n g(t)\} = (-1)^n \frac{d^n}{ds^n}(\mathcal{L}(g(t)))$ we get

$$(-1)^1 \frac{d}{ds}[sY(s) - 1] + Y(s) = \frac{s}{s^2-1}$$

Solving for $Y'(s)$

$$Y'(s) = -\frac{1}{s^2-1}$$

Integrating and ignoring the constant of integration yields

$$Y(s) = -\frac{1}{2}\ln\left(\frac{s-1}{s+1}\right).$$

Now $y(t) = \mathcal{L}^{-1}(Y(s)) = -\frac{1}{2}\mathcal{L}^{-1}\left\{\ln\left(\frac{s-1}{s+1}\right)\right\}$.

Using $\mathcal{L}^{-1}\left\{\ln\left(\frac{s-a}{s-b}\right)\right\} = \frac{1}{t}(e^{bt} - e^{at})$

we get

$$y(t) = -\frac{1}{2t}(e^{-t} - e^{t})$$

$$= \frac{1}{t}\sinh t$$

4-70

Solve the differential equation

$$y'' + 2y' + y = \delta(t - 2) \quad y(0) = 1,\ y'(0) = 0$$

where δ is the Dirac "Delta function."

****

$$\mathcal{L}\left(y'' + 2y' + y = \delta(t-2)\right)$$

$$\mathcal{L}y'' + 2\mathcal{L}y' + \mathcal{L}y = \mathcal{L}\delta(t-2)$$

$$s^2Y(s) - s\underbrace{y(0)}_{1} - \underbrace{y'(0)}_{0} + 2\left\{sY(s) - \underbrace{y(0)}_{1}\right\} + Y(s) = e^{-2s}$$

$$\Rightarrow \left[s^2+2s+1\right]Y(s) = e^{-2s} + s + 2$$

$$\therefore Y(s) = e^{-2s}\cdot\frac{1}{(s+1)^2} + \frac{s+2}{(s+1)^2}$$

$$= e^{-2s}\cdot\frac{1}{(s+1)^2} + \frac{(s+1)+1}{(s+1)^2}$$

$$\therefore Y(s) = e^{-2s}\cdot\frac{1}{(s+1)^2} + \frac{1}{(s+1)} + \frac{1}{(s+1)^2}$$

$$y(t) = \mathcal{L}^{-1}Y(s) = \mathcal{L}^{-1}\left\{e^{-2s}\cdot\frac{1}{(s+1)^2}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2}\right\} \quad **$$

$F(s)$

$$\mathcal{L}^{-1}\left\{e^{-2s}\cdot\frac{1}{(s+1)^2}\right\} = \mathcal{L}^{-1}\left\{e^{-2s}\cdot F(s)\right\} = *$$

e^{-t} $\qquad$ $f(t) = te^{-t}$

$F(s)$, $f(t) = \mathcal{L}^{-1}F(s)$

$* = u_2(t)\,f(t-2)$ where $u_c = \begin{cases} 0 & t < c \\ 1 & t \geq c \end{cases}$

But $\mathcal{L}^{-1}F(s) = \mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2}\right\} = \mathcal{L}^{-1}\left\{G(s+1)\right\} = e^{-t}g(t) = e^{-t}\cdot t = f(t)$

$\left[\text{where } G(s) = \frac{1}{s^2} \text{ and } g(t) = t\right]$

$\therefore f(t) = te^{-t}$

$\therefore \mathcal{L}^{-1}\left\{e^{-2s}\cdot\frac{1}{(s+1)^2}\right\} = u_2(t)(t-2)e^{-(t-2)}$; Hence, ** can how be rewritten as: $y(t) = u_2(t)(t-2)e^{-(t-2)} + e^{-t} + te^{-t}$

4-73

Solve the following second order differential equation by Laplace Transform

$$y''(t) + 4\,y'(t) + 4\,y(t) = \delta(t) - \delta(t-1)$$

with initial conditions: $y(0) = 1$ and $y'(0) = 2$.

****

Take the Laplace Transform on both sides of the equation and use the formulae

$$\mathcal{L}\{y'(t)\} = sY(s) - y(0), \quad \mathcal{L}\{\delta(t-1)\} = e^{-s}, \quad \mathcal{L}\{\delta(t)\} = 1$$

and $\mathcal{L}\{y''(t)\} = s^2Y(s) - sy(0) - y'(0)$. We have

$$(s^2+4s+4)\,Y(s) = 1 - e^{-s} + s + 6.$$

We write $Y(s)$ in the form

$$Y(s) = \frac{5 - e^{-s}}{(s+2)^2} + \frac{s+2}{(s+2)^2}$$

$$= \frac{1}{s+2} + \frac{5}{(s+2)^2} - \frac{e^{-s}}{(s+2)^2}.$$

Taking the inverse Laplace Transform

$$y(t) = e^{-2t} + 5t\,e^{-2t} - u_1(t)(t-1)\,e^{-2(t-1)}.$$

4-72

Solve the following differential equation and graph the solution.

$$y'' + y = 2 \cdot \delta(t - 2\pi) - 2 \cdot \delta(t - 4\pi)$$

$$y(0) = 0, \quad y'(0) = 0$$

****

$$\mathcal{L}\{y'' + y = 2\delta(t-2\pi) - 2\delta(t-4\pi)\}$$

$$\mathcal{L}y'' + \mathcal{L}y = 2\mathcal{L}\delta(t-2\pi) - 2\mathcal{L}\delta(t-4\pi)$$

$$s^2Y(s) - \overset{0}{y'(0)}\,s - \overset{0}{y'(0)} + Y(s) = 2e^{-2\pi s} - 2e^{-4\pi s}$$

$$\Rightarrow [s^2+1]\,Y(s) = 2e^{-2\pi s} - 2e^{-4\pi s} \Rightarrow Y(s) = 2e^{-2\pi s}\cdot\frac{1}{s^2+1} - 2e^{-4\pi s}\cdot\frac{1}{s^2+1}$$

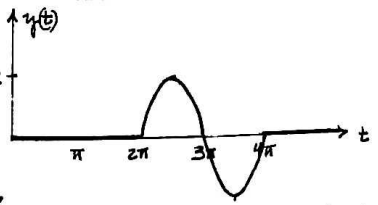
$$\text{Hence, } y(t) = \mathcal{L}^{-1}\{Y(s)\} = 2\mathcal{L}^{-1}\left\{e^{-2\pi s}\cdot\frac{1}{s^2+1}\right\} - 2\mathcal{L}^{-1}\left\{e^{-4\pi s}\cdot\frac{1}{s^2+1}\right\}$$

$$F(s) \Leftrightarrow f(t) = \sin t \Leftrightarrow F(s)$$

$$\therefore y(t) = 2u_{2\pi}\sin(t-2\pi) - 2u_{4\pi}\sin(t-4\pi)$$
$$= 2u_{2\pi}\sin t - 2u_{4\pi}\sin t$$

$$\therefore y(t) = 2\sin t\,[u_{2\pi} - u_{4\pi}]$$ (where u_c =

Graphing:



Note: This solution could describe the action of a suitable simple spring and weight (with no friction) with an impulsive force down at $t=2\pi$ and an equal and opposite impulsive force up at $t=4\pi$.

4-75

$$\frac{dy}{dt} + y = 1, \quad 0 \le t < 1$$
$$= 0, \quad t \ge 1 \qquad y(0) = 0.$$

$y(t) =$ (A) $H(t) - e^{-t} + e^{-t}H(t-1) - H(t-1)$

(B) $H(t) - e^{-t} + e^{-(t-1)}H(t-1) - H(t-1)$

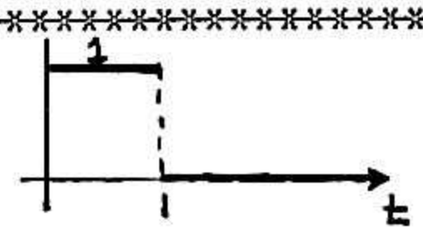
(C) $1 - e^{-t}$

(D) $e^{-t}H(t-1) - H(t-1)$

(E) $1 - e^{-(t-1)} + e^{-(t-1)}H(t-1) - H(t-1)$

****

$g(t) = 1, \ 0 \le t < 1$ is the "pulse"
$= 0, \ t \ge 1$



It is obtained by subtracting $H(t-1)$ from $H(t)$, where $H(t)$ is the Heaviside function that is 1 for any $t>0$, while $H(t-1)$ is zero before $t=1$ and 1 after $t=1$. We have

$$g(t) = H(t) - H(t-1).$$

We use the formula

$$(\bigstar) \quad f(t-1)\,H(t-1) \longrightarrow F(s)\,e^{-s} \quad (\text{where } f(t) \to F(s)).$$

To transform $H(t)$, note that $f(t) = 1 \to \frac{1}{s} = F(s)$. Hence by $(\bigstar)$.

$$H(t) \longrightarrow \frac{1}{s}\,e^{0\cdot s} = \frac{1}{s}.$$

To transform $H(t-1)$, note that $f(t-1) = 1$ that is f is identically one and by $(\bigstar)$

$$H(t-1) \to \frac{1}{s}\,e^{-s}$$

Transforming the equation

$$\underset{\downarrow}{\frac{dy}{dt}} + \underset{\downarrow Y(s)}{y} = \underset{\downarrow \frac{1}{s}}{H(t)} - \underset{\downarrow \frac{1}{s}e^{-s}}{H(t-1)}$$
$$sY(s) - y(0)$$

4-74

Solve this equation with the Laplace transform:
$y''(t)-2y'(t)+y(t) = 3\cos(2t)+2e^t$; t=0, y(0)=0, y'(0)=0.

****

Solution: $L\{y''-2y'+y\} = L\{3\cos 2t + 2e^t\}$

$\Rightarrow L\{y''\} - 2L\{y'\} + L\{y\} = 3L\{\cos 2t\} + 2L\{e^t\}$

$\Rightarrow s^2 f(s) - s\,y'(0) - y'(0) - 2[s f(s) - y(0)] + f(s)$

$= 3\frac{s}{s^2+4} + 2\frac{1}{s-1}$ where $L\{y\} = f(s)$.

$\Rightarrow s^2 f(s) - 2s f(s) + f(s) = \frac{3s}{s^2+4} + \frac{2}{s-1}$

$\Rightarrow (s^2-2s+1) f(s) = \frac{3s}{s^2+4} + \frac{2}{s-1} = (s-1)^2 f(s)$

$\Rightarrow f(s) = \frac{1}{(s-1)^2}\left[\frac{3s}{s^2+4} + \frac{2}{s-1}\right]$

$\Rightarrow L^{-1}\{f(s)\} = y(t) = L^{-1}\left\{\frac{1}{(s-1)^2}\left[\frac{3s}{s^2+4} + \frac{2}{s-1}\right]\right\}$

$= e^t L^{-1}\left\{\frac{1}{(s+1-1)^2}\left[\frac{3(s+1)}{(s+1)^2+4} + \frac{2}{s+1-1}\right]\right\} = e^t\left\{\frac{3s+3}{s^2(s^2+2s+5)} + \frac{2}{s^3}\right\}$

$= e^t\left[L^{-1}\left\{\frac{2}{s^3}\right\} + L^{-1}\left\{\frac{9/25}{s}\right\} + L^{-1}\left\{\frac{3/5}{s^2}\right\} + L^{-1}\left\{\frac{-33/25 - 9s/25}{s^2+2s+5}\right\}\right]$

$\Rightarrow e^t\left[t^2 + \frac{3}{5}t + \frac{9}{25} + \frac{e^{-t}}{25} L^{-1}\left\{\frac{-33-9(s-1)}{(s-1+1)^2+4}\right\}\right]$ where $s^2+2s+5 = (s+1)^2+4$.

$= e^t\left[t^2 + \frac{3}{5}t + \frac{9}{25} + \frac{e^{-t}}{25} L^{-1}\left\{\frac{-9s}{s^2+4} - 12\frac{2}{s^2+4}\right\}\right]$

$= e^t\left[t^2 + \frac{3}{5}t + \frac{9}{25}\right] - \frac{9}{25}\cos 2t - \frac{12}{25}\sin 2t = y(t)$

4-76

Use Laplace transform methods to solve the initial value problem

$$y'' - 3y' - 4y = e^t \; ; \; y(0) = 0 \text{ and } y'(0) = 1.$$

****

Take Laplace transforms of both sides of the differential equation to obtain

$$s^2 Y(s) - s\,y(0) - y'(0) - 3\,(sY(s) - y(0)) - 4\,Y(s) = \frac{1}{s-1}$$

i.e. $$s^2 Y(s) - 1 - 3sY(s) - 4\,Y(s) = \frac{1}{s-1}$$

i.e. $$Y(s) = \left(1 + \frac{1}{s-1}\right) \Big/ (s^2 - 3s - 4)$$

$$= \frac{s}{(s-1)(s+1)(s-4)}$$

Let $\dfrac{s}{(s-1)(s+1)(s-4)} \equiv \dfrac{A}{s-1} + \dfrac{B}{s+1} + \dfrac{C}{(s-4)}$. Then

$$s \equiv A(s+1)(s-4) + B(s-1)(s-4) + C(s-1)(s+1)$$

$s=1$: $\quad 1 = -6A$, i.e. $A = -\frac{1}{6}$

$s=-1$: $\quad -1 = 10B$, i.e. $B = -\frac{1}{10}$

$s=4$: $\quad 4 = 15C$, i.e. $C = \frac{4}{15}$

$$\therefore \; Y(s) = -\frac{1}{6}\frac{1}{s-1} - \frac{1}{10}\frac{1}{s+1} + \frac{4}{15}\frac{1}{s-4} \quad \text{and}$$

$$y(t) = -\frac{1}{6}e^t - \frac{1}{10}e^{-t} + \frac{4}{15}e^{4t}.$$

We obtain $(S+1)Y = \frac{1}{S} - \frac{1}{S}e^{-s}$

Solving for $Y(s)$: $Y(s) = \frac{1}{(S+1)S} - \frac{1}{(S+1)S}e^{-s}$.

To transform back to t, we decompose $1/(S+1)S$ into partial fractions

$$\frac{1}{(S+1)S} = \frac{A}{S} + \frac{B}{S+1} = \frac{A(S+1)+BS}{(S+1)S}$$

$$1 = A(S+1) + BS$$

$S=0 \Rightarrow 1 = A$

$S=-1 \Rightarrow 1 = B(-1)$; Hence $\frac{1}{(S+1)S} = \frac{1}{S} - \frac{1}{S+1}$.

$$Y(s) = \frac{1}{S} - \frac{1}{S+1} - \left(\frac{1}{S} - \frac{1}{S+1}\right)e^{-s}$$

$$= \underset{\downarrow}{\frac{1}{S}} - \underset{\downarrow}{\frac{1}{S+1}} - \frac{1}{S}e^{-s} + \frac{1}{S+1}e^{-s}$$

$$1 \qquad e^{-t}$$

By formula (★)

$$\underbrace{\frac{1}{S}}_{\downarrow}e^{-s} \longrightarrow H(t-1)$$

$1 = f(t)$; we need $f(t-1)$ which is 1.

$$\underbrace{\frac{1}{S+1}}_{\downarrow}e^{-s} \longrightarrow \underbrace{e^{-(t-1)}}_{\uparrow} H(t-1).$$

$e^{-t} = f(t)$; we need $f(t-1)$

In conclusion,

$$y(t) = 1 - e^{-t} - H(t-1) + e^{-(t-1)}H(t-1).$$

The function 1 and the function $H(t)$ are the same since we consider $f(t)$ only for time $t > 0$. Hence, the correct answer is (B).

4-78

Solve the third order initial value problem for y(t) using the method of Laplace transforms.

$$y''' + y'' + 3y' - 5y = 16e^{-t}; \quad y(0) = 0, \; y'(0) = 2, \; y''(0) = -4$$

****

Let $Y(s) = \mathcal{L}\{y\}$. Then taking the Laplace Transform of both sides of the equation gives

$$\mathcal{L}\{y'''\} + \mathcal{L}\{y''\} + 3\mathcal{L}\{y'\} - 5\,Y(s) = 16/(s+1)$$

Using intial conditions and properties of $\mathcal{L}$, we have

$$\mathcal{L}\{y'''\} = s^3 Y(s) - s^2 y(0) - s y'(0) - y''(0) = s^3 Y(s) - 2s + 4$$

$$\mathcal{L}\{y''\} = s^2 Y(s) - s\,y(0) - y'(0) = s^2 Y(s) - 2$$

$$\mathcal{L}\{y'\} = s\,Y(s) - y(0) = s\,Y(s)$$

Substitution and simplification gives

$$s^3 Y(s) + s^2 Y(s) + 3s\,Y(s) - 5\,Y(s) = 16/(s+1) + 2s - 2$$

So $$Y(s) = \frac{16}{(s+1)(s^3 + s^2 + 3s - s)} + \frac{2(s-1)}{(s^3 + s^2 + 3s - s)}$$

$$= \frac{16}{(s+1)(s-1)(s^2 + 2s + 5)} + \frac{2}{(s^2 + 2s + 5)}$$

$$\vdots \quad \text{(By partial fraction technigues.)}$$

$$= \frac{-2}{s+1} + \frac{1}{s-1} + \frac{s+1}{s^2 + 2s + 5}$$

4-77

Solve, using Laplace Transform: $y'' + 4y' + 4y = te^{-t}$;

$y(0) = y'(0) = 0.$

****

$L\{y'' + 4y' + 4y\} = L\{te^{-t}\}$. Let $L\{y\} = Y(s)$. Then

$s^2 Y(s) - s\,y(0) - y'(0) + 4sY(s) - 4y(0) + 4Y(s) = -\frac{d}{ds}\frac{1}{s+1}$

$s^2 Y(s) + 4sY(s) + 4Y(s) = \frac{1}{(s+1)^2}$.

$Y(s)(s^2+4s+4) = \frac{1}{(s+1)^2}$. Then $Y(s) = \frac{1}{(s+1)^2(s+2)^2}$.

$\frac{1}{(s+1)^2(s+2)^2} = \frac{A}{(s+1)^2} + \frac{B}{s+1} + \frac{C}{(s+2)^2} + \frac{D}{s+2}$ from algebra.

$\therefore 1 = A(s+2)^2 + B(s+1)(s+2)^2 + C(s+1)^2 + D(s+2)(s+1)^2$

using $s=-2$, $1 = 0 + 0 + C(-2+1)^2 + 0$ or $C = 1$.

using $s=-1$, $1 = A(-1+2)^2 + 0 + 0 + 0$ or $A = 1$.

$\therefore 1 = (s+2)^2 + B(s+1)(s+2)^2 + (s+1)^2 + D(s+2)(s+1)^2$

using $s=0$, $1 = 4 + 4B + 1 + 2D$ or $4B + 2D = -4$

using $s=1$, $1 = 9 + 18B + 4 + 12D$ or $3B + 2D = -2$

solving, $B = -2$ and $D = 2$

$\therefore Y(s) = \frac{1}{(s+1)^2} + \frac{-2}{s+1} + \frac{1}{(s+2)^2} + \frac{2}{s+2}$.

$y(t) = L^{-1}\left\{\frac{1}{(s+1)^2} + \frac{-2}{s+1} + \frac{1}{(s+2)^2} + \frac{2}{s+2}\right\}$

$= te^{-t} - 2e^{-t} + te^{-2t} + 2e^{-2t}$.

Note: $L\{t\,f(t)\} = -\frac{d}{ds}L\{f(t)\}$ and $L^{-1}\{F(s+a)\} = e^{-at}L^{-1}\{F(s)\}$.

4-80

Solve the following initial value problem using Laplace transforms.

$$y'' - 4y' + 4y = 3te^{2t},$$

$$y(0) = 1, \quad y'(0) = 2.$$

* *

Take Laplace transform of each term of the equation and write

$$\mathcal{L}[y''] - 4\mathcal{L}[y'] + 4\mathcal{L}[y] = \mathcal{L}[3te^{2t}].$$

This gives

$$[s^2\bar{y}(s) - sy(0) - y'(0)] - 4[s\bar{y}(s) - y(0)] + 4\bar{y}(s) = \frac{3}{(s-2)^2}.$$

Here we have used the notation $\mathcal{L}[y(t)] = \bar{y}(s)$.

Inserting the initial conditions and simplifying, we obtain

$$(s^2 - 4s + 4)\bar{y}(s) = s - 2 + \frac{3}{(s-2)^2}$$

so that

$$\bar{y}(s) = \frac{1}{s-2} + \frac{3}{(s-2)^4}.$$

Therefore

$$y(t) = \mathcal{L}^{-1}\left\{\frac{1}{s-2}\right\} + \mathcal{L}^{-1}\left\{\frac{3}{(s-2)^4}\right\}.$$

Using the translation theorem and a Laplace transform table, the second term is found to be $\frac{1}{2}t^3e^{2t}$. As a result, we have

$$y(t) = e^{2t} + \frac{1}{2}t^3e^{2t} = (1 + \frac{1}{2}t^3)e^{2t}.$$

Thus $y(t) = \mathcal{L}^{-1}\{Y(s)\}$

$$= \mathcal{L}^{-1}\left\{\frac{-2}{s+1}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\} + \mathcal{L}^{-1}\left\{\frac{s+1}{(s+1)^2+2^2}\right\}$$

$$= \underline{\underline{-2e^{-t} + e^{t} + e^{-t}\cos 2t}}$$

4-79

Obtain the Laplace transform of the solution to the differential equation $y'' + y = 16 \cos t$, $y(0) = 0$, $y'(0) = 0$.

****

let $\mathcal{L}\{y(t)\} = y(s)$, then

$\mathcal{L}\{y'' + y = 16\cos t\}$ yields

$\mathcal{L}\{y''\} + \mathcal{L}\{y\} = 16\mathcal{L}\{\cos t\}$

$s^2y(s) - sy'(0) - y(0) + y(s) = 16\frac{s}{s^2+1}$

$s^2y(s) + y(s) = \frac{16s}{s^2+1}$

$y(s)(s^2+1) = \frac{16s}{s^2+1}$

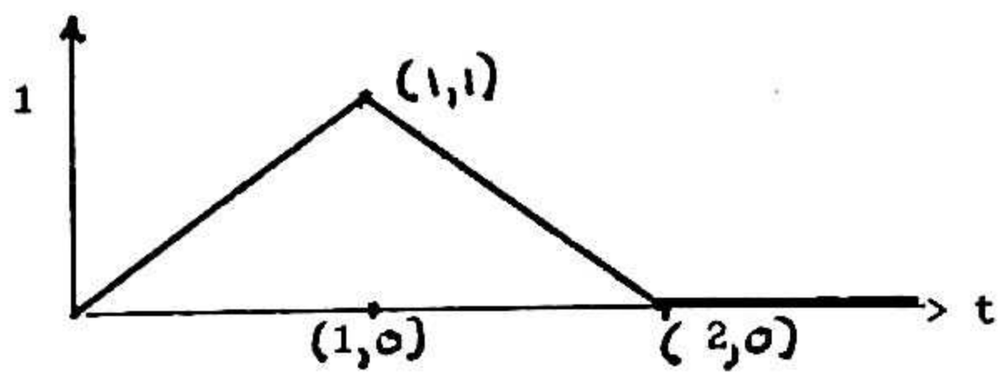
$\underline{\underline{y(s) = \frac{16s}{(s^2+1)^2}}}$

4-82

Solve the differential equation

$$y'' + y = f(t) \quad \text{where} \quad f(t) = \begin{cases} t & 0 \le t < 1 \\ -t + 2 & 1 \le t < 2 \\ 0 & t \ge 2 \end{cases}$$

$$y(0) = 0,\ y'(0) = 0$$



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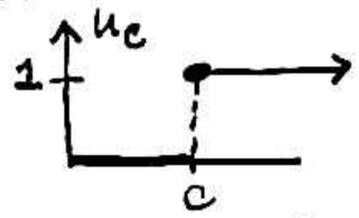
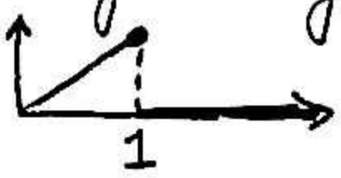
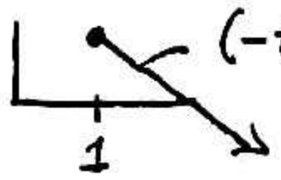
If $u_c(t) = \begin{cases} 0 & t < c \\ 1 & t \ge c \end{cases}$ 

, $\mathcal{L}\{u_c\} = \frac{e^{-cs}}{s}$, $s > 0$

First we write f in terms of u_c functions:

Note: $f(t) = t$ is O.K. up to $t = 1$. From $t \ge 1$ we need to subtract off t — by using $t\,u_1$

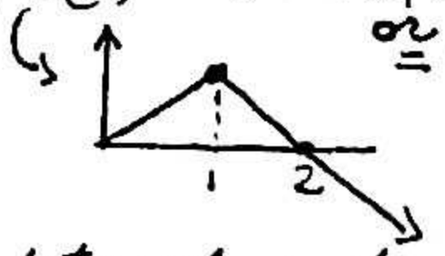
$\therefore f(t) = t - t\,u_1$ is  ; Now we need to



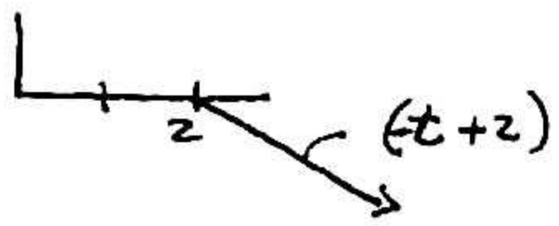
add

O.K. out to $t = 2$

So, $f(t) = t - t\,u_1 + (-t+2)\,u_1$ this is now or $= t + (2-2t)\,u_1$



We next need to subtract out from $t = 2$ on, i.e., we need to subtract $-(-t+2)\,u_2$



$\therefore$ $\underline{f(t) = t + (2-2t)\,u_1 + (t-2)\,u_2}$

Then $y'' + y = t + (2-2t)\,u_1 + (t-2)\,u_2$

$\mathcal{L}$ $\mathcal{L}y'' + \mathcal{L}y = \mathcal{L}t + \mathcal{L}\{(2-2t)\,u_1\} + \mathcal{L}\{(t-2)\,u_2\}$

$= \frac{1}{s^2} + \mathcal{L}\{u_1 \cdot g(t-1)\} + \mathcal{L}\{u_2 \cdot h(t-2)\}$

where $g(t) = -2t$ where $h(t) = t$

$s^2Y(s) - s\overset{0}{y}(0) - \overset{0}{y'}(0) + Y(s) = \frac{1}{s^2} + e^{-s}\left(-\frac{2}{s^2}\right) + e^{-2s}\left(\frac{1}{s^2}\right)$

4-81

Solve $y''+2y'+5y=2e^{-4t}$, $y(0)=1, y'(0)=0$.

****

Taking the Laplace transform of the entire equation,

$s^2Y(s) - s + 2(sY(s)-1) + 5Y(s) = \frac{2}{s+4}$, $Y(s) = \mathcal{L}\{y(t)\}$

$(s^2+2s+5)\,Y(s) = s+2+\frac{2}{s+4}$, so

$Y(s) = \frac{s+2}{s^2+2s+5} + \frac{2}{(s+4)(s^2+2s+5)}$. Do partial fractions:

$$\frac{2}{(s+4)(s^2+2s+5)} = \frac{A}{s+4} + \frac{Bs+C}{s^2+2s+5}$$

$2 = A(s^2+2s+5) + (Bs+C)(s+4)$: $s=-4 \to 2=13A$, $A=\frac{2}{13}$

$s=0 \to 2=5A+4C$, $C=\frac{4}{13}$

$s=-3 \to 2=8A-3B+C$, $B=-\frac{2}{13}$

Then

$$Y(s) = \frac{s+2}{s^2+2s+5} + \frac{2/13}{s+4} + \frac{-\frac{2}{13}s+\frac{4}{13}}{s^2+2s+5}$$

$$= \frac{2/13}{s+4} + \frac{\frac{11}{13}s+\frac{30}{13}}{s^2+2s+5}$$

Now, $s^2+2s+5 = (s+1)^2+4$

$$= \frac{2/13}{s+4} + \frac{11}{13}\,\frac{s+\frac{30}{11}}{(s+1)^2+4} = \frac{2/13}{s+4} + \frac{11}{13}\,\frac{s+1+\frac{19}{11}}{(s+1)^2+4}$$

$$Y(s) = \frac{2}{13}\frac{1}{s+4} + \frac{11}{13}\frac{s+1}{(s+1)^2+4} + \frac{19}{26}\frac{2}{(s+1)^2+4}$$

$$y(t) = \frac{2}{13}e^{-4t} + \frac{11}{13}e^{-t}\cos 2t + \frac{19}{26}e^{-t}\sin 2t$$

4-83

Solve the initial value problem

$$x' + x = \delta(t - 1)$$

$$x(0) = 0$$

using Laplace transforms. Sketch the solution.

****

Laplace transform the equation

$$\mathcal{L}\{x'\} + \mathcal{L}\{x\} = \mathcal{L}\{\delta(t-1)\}$$

$$\Rightarrow \quad sX(s) - x(0) + X(s) = e^{-s}$$

$$(s+1)X(s) = e^{-s}, \text{ since } x(0) = 0$$

$$X(s) = \frac{e^{-s}}{s+1}$$

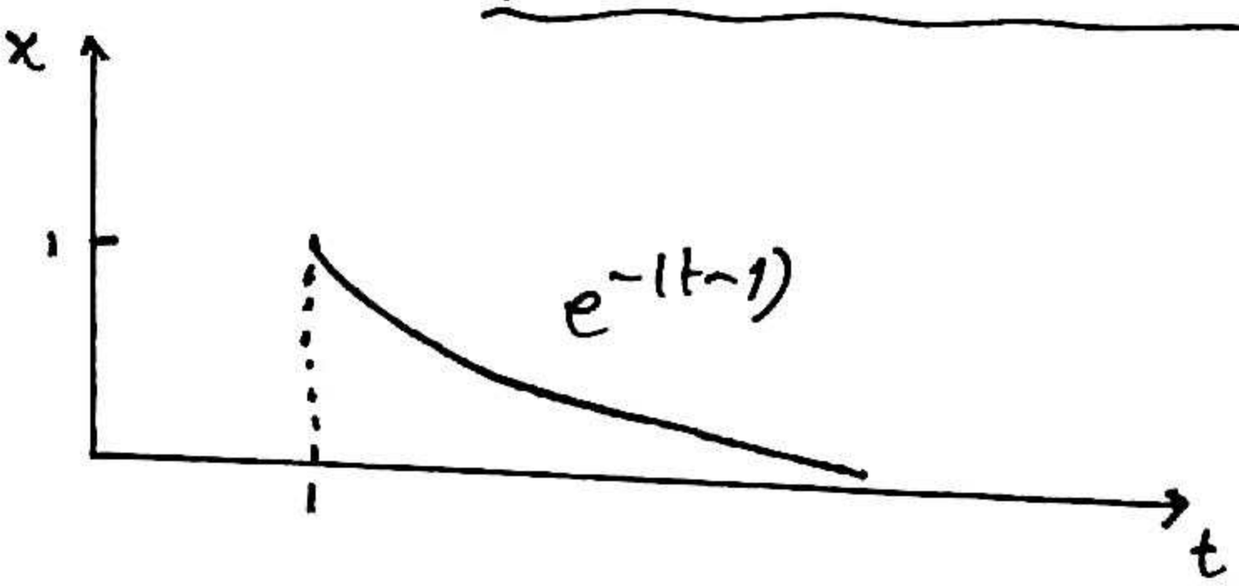
By the shift theorem on the t-axis,

$$\mathcal{L}^{-1}\{e^{-as}F(s)\} = f(t-a)u(t-a),$$

where $u(t-a) = \begin{cases} 0, & t < a \\ 1, & t > a \end{cases}$

Here $F(s) = \frac{1}{s+1}$, $a = 1$, so

$$\mathcal{L}^{-1}\{X(s)\} = \boxed{x(t) = e^{-(t-1)}u(t-1)}$$



$$\Rightarrow [s^2+1]\,Y(s) = \frac{1}{s^2} - 2e^{-s}\left(\frac{1}{s^2}\right) + e^{-2s}\left(\frac{1}{s^2}\right)$$

$$\therefore \quad Y(s) = \frac{1}{s^2(s^2+1)} - 2e^{-s}\cdot\frac{1}{s^2(s^2+1)} + e^{-2s}\cdot\frac{1}{s^2(s^2+1)}$$

But, using partial fractions: $\dfrac{1}{s^2(s^2+1)} = \dfrac{1}{s^2} + \dfrac{-1}{s^2+1}$

$$\therefore Y(s) = \left\{\frac{1}{s^2} - \frac{1}{s^2+1}\right\} - 2e^{-s}\cdot\frac{1}{s^2} + 2e^{-s}\cdot\frac{1}{s^2+1} + e^{-2s}\cdot\frac{1}{s^2} - e^{-2s}\cdot\frac{1}{s^2+1}$$

$$\therefore y(t) = \mathcal{L}^{-1}Y(s) = \mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\} - 2\mathcal{L}^{-1}\left\{e^{-s}\cdot\frac{1}{s^2}\right\} + 2\mathcal{L}^{-1}\left\{e^{-s}\frac{1}{s^2+1}\right\}$$

$$+ \mathcal{L}^{-1}\left\{e^{-2s}\cdot\frac{1}{s^2}\right\} - \mathcal{L}^{-1}\left\{e^{-2s}\cdot\frac{1}{s^2+1}\right\}$$

(in order) ... etc..

$$\therefore y(t) = t - \sin t - 2u_1(t)(t-1) + 2u_1(t)\sin(t-1) + u_2(t)(t-2) - u_2(t)\sin(t-2)$$

$$y(t) = t - \sin t + 2u_1(t)\left[1 - t + \sin(t-1)\right] + u_2(t)\left[t-2-\sin(t-2)\right]$$

or, equivalently:

$$y(t) = \begin{cases} t - \sin t & 0 \le t < 1 \\ 2 - t - \sin t + 2\sin(t-1) & 1 \le t < 2 \\ -\sin t + 2\sin(t-1) - \sin(t-2) & 2 \le t \end{cases}$$

4-85

Use Laplace Transforms to solve the initial value problem

$y'' - 4y' + 4y = t, \qquad y(0) = A, \qquad y'(0) = B.$

****

Apply the Laplace transform to the differential equation.

$$\mathcal{L}(y'') - 4\mathcal{L}(y') + 4\mathcal{L}(y) = \mathcal{L}(t).$$

Thus $s^2\mathcal{L}(y) - sA - B - 4[s\mathcal{L}(y) - A] + 4\mathcal{L}(y) = \frac{1}{s^2}$. Therefore $(s^2 - 4s + 4)\mathcal{L}(y) = \frac{1}{s^2} + As + B + 4A$. Thus

$$(s-2)^2\mathcal{L}(y) = \frac{1}{s^2} + As + B + 4A$$

or $$\mathcal{L}(y) = \frac{1}{s^2(s-2)^2} + \frac{As + B + 4A}{(s-2)^2}.$$

Apply partial fractions to write

$$\frac{1}{s^2(s-2)^2} = \frac{C}{s} + \frac{D}{s^2} + \frac{E}{s-2} + \frac{E}{(s-2)^2}.$$

Multiply by $s^2(s-2)^2$ to obtain

$$1 = Cs(s-2)^2 + D(s-2)^2 + Es^2(s-2) + Fs^2$$

Let $s = 0$ to obtain $1 = 4D$ or $D = 1/4$.

Let $s = 2$ to obtain $1 = 4F$ or $F = 1/4$.

Let $s = 1$ to obtain $1 = C + D - E + F$.

This gives $\frac{1}{2} = C - E$.

4-84

Solve $y'' + 25y = 40\sin 5t$ with initial conditions $y(0) = y'(0) = 0$.

****

Let $\bar{y} = L[y(t)]$. Transforming the equation, $s^2\bar{y} + 25\bar{y} = 40\frac{5}{s^2+25}$. Solving for $\bar{y}$, we have $\bar{y} = \frac{40 \cdot 5}{(s^2+25)^2}$.

To invert this, we apply Table Entry #18 on p. 246 of Schaum's Outline Series on Laplace Transforms, which says

$$L^{-1}\left[\frac{1}{(s^2+a^2)^2}\right] = \frac{\sin at - at\cos at}{2a^3}.$$

With $a = 5$, we have then the solution

$$y = \frac{200}{250}(\sin 5t - 5t\cos 5t)$$

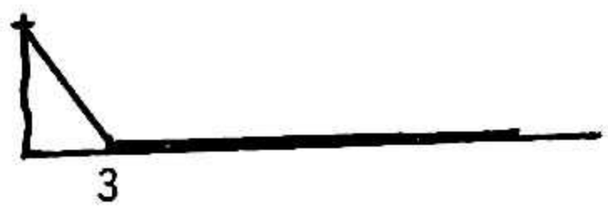
$$= \frac{4}{5}\sin 5t - 4t\cos 5t.$$

Note that the second term is what is commonly called a "resonant" term, because its amplitude increases without bound as $t \to \infty$.

4-86

Solve for y.

$$y'' + 7y' + 12y = r(x),\ y(0) = 1,\ y'(0) = -1$$

$r(x) =$ 

****

Laplace transforms give:

$$p^2 Y(p) - p \cdot y(0) - y'(0) + 7\left(pY(p) - y(0)\right) + 12Y(p) = R(p) \quad \text{or}$$

$$p^2 Y - p + 1 + 7pY - 7 + 12Y = R$$

$$(p^2 + 7p + 12)Y - p - 6 = R.$$

Now we need R, the transform of r(x):

First write r(x) using Heaviside functions; there is a change at 3, so:

$$r(x) = (1^{st}\ \text{stuff}) + H(x-3)(2^{nd}\ \text{stuff})$$

For $x < 3$, $H(x-3) = 0$ so $r = 1^{st}$ stuff.

We want $r = 6 - 2x$, so 1st stuff $= 6 - 2x$.

For $x > 3$, $H(x-3) = 1$, so $r = 6 - 2x + 2^{nd}$ stuff.

We want 0, so 2nd stuff $= 2x - 6$.

Thus $r(x) = 6 - 2x + H(x-3)(2x-6)$.

Let $s = -1$ to obtain $1 = -9C + 9D - 3E + F$.

This gives $\frac{1}{2} = 3C + E$.

Solving these two equations gives $C = 1/4$ and $E = -1/4$. Therefore $\frac{1}{s^2(s-2)^2} = \frac{1/4}{s} + \frac{1/4}{s^2} - \frac{1/4}{s-2} + \frac{1/4}{(s-2)^2}$.

Similarly, $\frac{As + 4A + B}{(s-2)^2} = \frac{\frac{A}{2}}{s-2} + \frac{5A+B}{(s-2)^2}$.

Hence, $\mathcal{L}(y) = \frac{1}{4}\left[\frac{1}{s} + \frac{1}{s^2} - \frac{1}{s-2} + \frac{1}{(s-2)^2}\right] + \frac{A}{2} \cdot \frac{1}{s-2} + \frac{5A+B}{(s-2)^2}$.

Inverting this gives

$$y = \frac{1}{4}\left[1 + t - e^{2t} + te^{2t}\right] + \frac{A}{2}e^{2t} + (5A+B)te^{2t}$$

Therefore

$$y = \frac{1}{4} + \frac{1}{4}t + \left(\frac{A}{2} - \frac{1}{4}\right)e^{2t} + \left(5A + B + \frac{1}{4}\right)te^{2t},$$

4-87

Solve the initial value problem

$$y'' + 4y' + 4y = \begin{cases} t & 0 \le t \le 1 \\ e^{-2(t-1)} & t \ge 1 \end{cases} \qquad y(0)=1,\ y'(0)=0.$$

****

Using Laplace transforms:

$$y''+4y'+4y = t - t\,u(t-1) + e^{-2(t-1)}\,u(t-1)$$

where $u(t-1) = \begin{cases} 0 & 0 \le t < 1 \\ 1 & t \ge 1 \end{cases}$

$$y''+4y'+4y = t - (t-1)\,u(t-1) - u(t-1) + e^{-2(t-1)}\,u(t-1)$$

$$[s^2Y(s) - sy(0) - y'(0)] + 4[sY(s) - y(0)] + 4Y(s) = \frac{1}{s^2} - e^{-s}\frac{1}{s^2} - e^{-s}\frac{1}{s} + e^{-s}\frac{1}{s+2}$$

where $Y(s) = \mathcal{L}\{y(t)\}$

using $y(0)=1$, $y'(0)=0$:

$$s^2Y(s) - s + 4sY(s) - 4 + 4Y(s) = \frac{1}{s^2} - e^{-s}\frac{1}{s^2} - e^{-s}\frac{1}{s} + e^{-s}\frac{1}{s+2}$$

$$(s^2+4s+4)\,Y(s) = \frac{1}{s^2} - e^{-s}\frac{1}{s^2} - e^{-s}\frac{1}{s} + e^{-s}\frac{1}{s+2} + s + 4$$

$$Y(s) = \frac{1}{s^2(s+2)^2} - e^{-s}\frac{1}{s^2(s+2)^2} - e^{-s}\frac{1}{s(s+2)^2} + e^{-s}\frac{1}{(s+2)^3} + \frac{s}{(s+2)^2} + \frac{4}{(s+2)^2}$$

partial fractions decompositions:

$$\frac{1}{s^2(s+2)^2} = \frac{-1/4}{s} + \frac{1/4}{s^2} + \frac{1/4}{s+2} + \frac{1/4}{(s+2)^2}$$

$$\frac{1}{s(s+2)^2} = \frac{1/4}{s} + \frac{-1/4}{s+2} + \frac{-1/2}{(s+2)^2}$$

Now $\mathcal{L}\left(6-2x+H(x-3)(2x-6)\right) =$

$\mathcal{L}(6-2x)+\mathcal{L}(H(x-3)(2x-6)) =$

$\frac{6}{p}-\frac{2}{p^2}+e^{-3p}\mathcal{L}(2(x+3)-6)$

$=\frac{6}{p}-\frac{2}{p^2}+e^{-3p}\cdot\frac{2}{p^2}=\frac{6p-2+2e^{-3p}}{p^2}$.

Thus $(p^2+7p+12)Y = p+6+\frac{6p-2+2e^{-3p}}{p^2}$,

$Y=\frac{p^3+6p^2+6p-2+2e^{-3p}}{(p+3)(p+4)p^2}$

$\frac{p^3+6p^2+6p-2}{(p+3)(p+4)p^2}=\frac{43/72}{p}+\frac{-1/6}{p^2}+\frac{7/9}{p+3}+\frac{-3/8}{p+4}$

$\frac{2e^{-3p}}{(p+3)(p+4)p^2}=e^{-3p}\left(\frac{-7/72}{p}+\frac{1/6}{p^2}+\frac{2/9}{p+3}+\frac{-1/8}{p+4}\right)$

Thus

$Y=\mathcal{L}\left(\frac{43}{72}-\frac{1}{6}x+\frac{7}{9}e^{-3x}-\frac{3}{8}e^{-4x}\right)+$

$e^{-3p}\mathcal{L}\left(\frac{-7}{72}+\frac{1}{6}x+\frac{2}{9}e^{-3x}-\frac{1}{8}e^{-4x}\right)$

$\Rightarrow y(x)=\frac{43}{72}-\frac{1}{6}x+\frac{7}{9}e^{-3x}-\frac{3}{8}e^{-4x}$

$+H(x-3)\left[\frac{-7}{72}+\frac{1}{6}(x-3)+\frac{2}{9}e^{-3(x-3)}-\frac{1}{8}e^{-4(x-3)}\right]$

MISCELLANEOUS RESULTS

4-88

Solve $y' + \int_0^t y(w)\,dw = 1$ with $y(0) = K_0$ a constant. Verify your results.

****

$$\mathcal{L}(y') + \mathcal{L}\left(\int_0^t y(w)\,dw\right) = \mathcal{L}(1)$$

$$sY(s) - y(0) + \frac{Y(s)}{s} = \frac{1}{s}$$

$$sY(s) - K_0 + \frac{Y(s)}{s} = \frac{1}{s}$$

$$Y(s)\left[\frac{s^2+1}{s}\right] = \frac{1}{s} + K_0$$

$$Y(s) = \frac{1}{s^2+1} + \frac{K_0 s}{s^2+1}$$

$$y(t) = \mathcal{L}^{-1}(Y(s)) = \mathcal{L}^{-1}\left(\frac{1}{s^2+1}\right) + K_0\,\mathcal{L}^{-1}\left(\frac{s}{s^2+1}\right)$$

$$\underline{y(t) = \sin t + K_0 \cos t}$$

$$\frac{d}{dt}(\sin t + K_0 \cos t) + \int_0^t [\sin(w) + K_0 \cos(w)]\,dw =$$

$$= \cos t - K_0 \sin t + (-\cos w)\Big|_0^t + K_0 \sin w\Big|_0^t =$$

$$= \cos t - K_0 \sin t - \cos t + 1 + K_0 \sin t + 0 = 1$$

$$\frac{s}{(s+2)^2} = \frac{1}{s+2} + \frac{-2}{(s+2)^2}$$

$$\text{so } Y(s) = \left[\frac{-1/4}{s} + \frac{1/4}{s^2} + \frac{1/4}{s+2} + \frac{1/4}{(s+2)^2}\right] - e^{-s}\left[\frac{-1/4}{s} + \frac{1/4}{s^2} + \frac{1/4}{s+2} + \frac{1/4}{(s+2)^2}\right]$$
$$- e^{-s}\left[\frac{1/4}{s} + \frac{-1/4}{s+2} + \frac{-1/2}{(s+2)^2}\right] + e^{-s}\frac{1}{(s+2)^2} + \frac{1}{s+2} + \frac{-2}{(s+2)^2}$$
$$+ \frac{4}{(s+2)^2}$$

$$Y(s) = \frac{-1/4}{s} + \frac{1/4}{s^2} + \frac{5/4}{s+2} + \frac{9/4}{(s+2)^2} + e^{-s}\left[\frac{-1/4}{s^2} + \frac{1/4}{(s+2)^2} + \frac{1}{(s+2)^3}\right]$$

$$\mathcal{L}^{-1}\left\{\frac{1}{(s+2)^2}\right\} = \mathcal{L}^{-1}\left\{-\frac{d}{ds}\left(\frac{1}{s+2}\right)\right\} = te^{-2t}$$

$$\mathcal{L}^{-1}\left\{\frac{1}{(s+2)^3}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{2}(-1)^2\frac{d^2}{ds^2}\left(\frac{1}{s+2}\right)\right\} = \frac{1}{2}t^2e^{-2t}$$

$$\text{so } y(t) = -1/4 + 1/4\,t + 5/4\,e^{-2t} + 9/4\,te^{-2t}$$
$$+ u(t-1)\left[-1/4\,(t-1) + 1/4\,(t-1)e^{-2(t-1)} + \frac{1}{2}(t-1)^2e^{-2(t-1)}\right]$$

4-90

Obtain the solution of the IVP for the system

$$\dot{x} = -x + y + 1$$

$$\dot{y} = x - y + 1$$

$$x(0) = y(0) = 0$$

by Laplace transforms

****

Laplace transform both equations

$$s\bar{X}(s) - x(0) = -\bar{X} + \bar{Y} + \frac{1}{s} \qquad x(0) = 0$$

$$s\bar{Y}(s) - y(0) = \bar{X} - \bar{Y} + \frac{1}{s} \qquad y(0) = 0$$

$$\Rightarrow (s+1)\bar{X}(s) - \bar{Y}(s) = \frac{1}{s}$$

$$-\bar{X}(s) + (s+1)\bar{Y}(s) = \frac{1}{s}$$

$$\bar{X}(s) = \frac{\begin{vmatrix} 1/s & -1 \\ 1/s & s+1 \end{vmatrix}}{\begin{vmatrix} s+1 & -1 \\ -1 & s+1 \end{vmatrix}} = \frac{\frac{1}{s}(s+1) + \frac{1}{s}}{(s+1)^2 - 1} = \frac{\frac{1}{s}(s+2)}{s(s+2)} = \frac{1}{s^2}$$

$$\bar{Y}(s) = \frac{\begin{vmatrix} s+1 & 1/s \\ -1 & 1/s \end{vmatrix}}{s(s+2)} = \frac{\frac{1}{s}(s+1) + \frac{1}{s}}{s(s+2)} = \frac{\frac{1}{s}(s+2)}{s(s+2)} = \frac{1}{s^2}$$

Thus $\boxed{x(t) = y(t) = \mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} = t}$

4-89

Solve the following system, using Laplace Transform:

$y' - z' - 2y + 2z = 1 - 2t; \quad y(0) = y'(0) = 0.$

$y'' + 2z' + y = 0; \quad z(0) = 0.$

****

$L\{y' - z' - 2y + 2z\} = L\{1 - 2t\}. \quad \therefore sY(s) - sZ(s) - 2Y(s) + 2Z(s) = \frac{1}{s} - \frac{2}{s^2}$

$L\{y'' + 2z' + y\} = L\{0\} \quad \therefore s^2Y(s) + 2sZ(s) + Y(s) = 0$

THEN $(s-2)Y(s) - (s-2)Z(s) = \frac{s-2}{s^2}$ OR $Y(s) - Z(s) = \frac{1}{s^2}, \; s > 2.$

FROM THE 2ND TRANSFORMED EQUATION, $(s^2+1)Y(s) + 2sZ(s) = 0.$

THIS IS A SYSTEM OF TWO LINEAR EQUATIONS IN $Y(s)$ AND $Z(s)$. MULTIPLY THE 1ST BY $2s$ AND ADD TO THE 2ND. WE OBTAIN, WITH PARTIAL FRACTIONS,

$$Y(s) = \frac{\frac{2}{s}}{s^2+2s+1} = \frac{2}{s(s+1)^2} = \frac{2}{s} - \frac{2}{s+1} - \frac{2}{(s+1)^2}.$$

$$Z(s) = Y(s) - \frac{1}{s^2} = \frac{2}{s} - \frac{2}{s+1} - \frac{2}{(s+1)^2} - \frac{1}{s^2}$$

$$\therefore y(t) = L^{-1}\left\{\frac{2}{s} - \frac{2}{s+1} - \frac{2}{(s+1)^2}\right\} = 2 - 2e^{-t} - 2te^{-t}$$

$$z(t) = L^{-1}\left\{Y(s) - \frac{1}{s^2}\right\} = 2 - 2e^{-t} - 2te^{-t} - t.$$

IN FINDING THESE INVERSE TRANSFORMS, WE USE THESE THEOREMS:

WITH n A POSITIVE INTEGER, $L^{-1}\left\{\frac{1}{s^n}\right\} = \frac{t^{n-1}}{(n-1)!}.$

THUS, E.G., $L^{-1}\left\{\frac{2}{s^2}\right\} = 2L^{-1}\left\{\frac{1}{s^2}\right\} = 2\frac{t^1}{1} = 2t.$

ALSO, $L^{-1}\{F(s+a)\} = e^{-at}L^{-1}\{F(s)\}.$

THUS, E.G., $L^{-1}\left\{\frac{2}{(s+1)^2}\right\} = 2e^{-t}L^{-1}\left\{\frac{1}{s^2}\right\} = 2te^{-t}.$

4-92

(a) If $F(s) = \mathcal{L}\{f(t)\}$ show that

$\mathcal{L}\{f(t) \cosh at\} = (1/2)[F(s-a) + F(s+a)]$.

(b) Use the result of part (a) to evaluate

$\mathcal{L}\{t \cosh at\}$

Simplify completely.

****

a) By definition,

$$\mathcal{L}\{f(t)\cosh at\} = \int_0^\infty e^{-st} f(t) \cosh at \, dt$$

$$= \int_0^\infty e^{-st} f(t) \frac{e^{at} + e^{-at}}{2} \, dt$$

$$= \frac{1}{2}\left[\int_0^\infty e^{-(s-a)t} f(t)\, dt + \int_0^\infty e^{-(s+a)t} f(t)\, dt\right]$$

$$= \frac{1}{2}\left[F(s-a) + F(s+a)\right]$$

as desired.

b) Since $\mathcal{L}\{t\} = 1/s^2$, we have by part (a) that

$$\mathcal{L}\{t\cosh at\} = \frac{1}{2}\left[\frac{1}{(s-a)^2} + \frac{1}{(s+a)^2}\right]$$

$$= \frac{s^2 + a^2}{(s^2 - a^2)^2}$$

4-91

Suppose a weight of 64 pounds is put on a spring with spring constant 2. The damping is negligible. The weight is released from rest at the equilibrium position. An impressed force f(t) = 2 acts on the system for 3 seconds and then is removed. Find the position at any time.

****

$w = 64$, $k = 2$, $\beta = 0$ $\quad x(0) = 0$, $v(0) = 0$

$f(t) = 2 - 2u(t-3)$.

$\frac{64}{32}x'' + 2x = 2 - 2u(t-3)$ or

$x'' + x = 1 - u(t-3)$ take the Laplace transform of both sides.

$s^2X(s) - s\cancel{x(0)}^{0} - \cancel{x'(0)}^{0} + X(s) = \frac{1}{s} - \frac{1}{s}e^{-3s}$

$(s^2+1)X(s) = \frac{1}{s} - \frac{1}{s}e^{-3s} = \frac{1}{s}(1 - e^{-3s})$

$X(s) = \frac{1}{s(s^2+1)}(1 - e^{-3s}) = \left[\frac{A}{s} + \frac{Bs+C}{s^2+1}\right](1 - e^{-3s})$

$A(s^2+1) + (Bs+C)s = 1$ $\quad s^2: A+B=0$

$As^2 + A + Bs^2 + Cs = 1$ $\quad s: C = 0$

const: $A = 1 \Rightarrow B = -1$

$X(s) = \left[\frac{1}{s} - \frac{s}{s^2+1}\right](1 - e^{-3s})$

Now take the inverse Laplace transform of both sides

$x(t) = 1 - \cos t - (1 - \cos(t-3))u(t-3)$.

4-95

used

$$\frac{dx}{dt} = y + e^{-t}$$

$$\frac{dy}{dt} = 4x + \sin t \qquad x(0) = y(0) = 0.$$

$x(t) =$ (A) $(13/60)e^{2t} - (11/60)e^{-2t} + (1/3)e^{-t} - (1/5)\sin t$

(B) $(13/60)e^{2t} - (11/20)e^{-2t} + (1/3)e^{-t} - (1/5)\sin t$

(C) $(13/20)e^{2t} - (11/60)e^{-2t} + (1/3)e^{-t} - (1/5)\cos t$

(D) $\frac{1}{2}e^{-t} + (1/3)\sin t - (13/11)\cos t$

(E) $(1/5)e^{2t} + (1/11)e^{-2t} + (1/3)\sin t$

****

The first equation transforms into $sX - \underbrace{x(0)}_{=0} = Y + \frac{1}{s+1}$

The second equation transforms into $sY - \underbrace{y(0)}_{=0} = 4X + \frac{1}{s^2+1}$

We rewrite these equations with X and Y on the same side

$$(I) \quad sX - Y = \frac{1}{s+1}$$

$$(II) \quad -4X + sY = \frac{1}{s^2+1}$$

We see that $(I) \times s + (II)$ will be an equation for X alone.

In fact,

$$\begin{array}{rl} & s^2X - sY = \frac{s}{s+1} \\ \oplus & -4X + sY = \frac{1}{s^2+1} \\ \hline & s^2X - 4X = \frac{s}{s+1} + \frac{1}{s^2+1} \end{array}$$

We obtain

$$(s^2-4)X = \frac{s(s^2+1) + s + 1}{(s+1)(s^2+1)}$$

$$X = \frac{s^3+2s+1}{(s^2-4)(s+1)(s^2+1)} = \frac{A}{s-2} + \frac{B}{s+2} + \frac{C}{s+1} + \frac{Ds+E}{s^2+1}$$

(real numbers)

by partial fraction decomposition.

4-93

Evaluate $\int_0^\infty t^2 e^{-3t} \sin t \, dt$ using Laplace transforms.

****

Notice that $\mathcal{L}\{t^2 \sin t\} = \int_0^\infty e^{-st} t^2 \sin t \, dt$

so $\int_0^\infty t^2 e^{-3t} \sin t \, dt = \mathcal{L}\{t^2 \sin t\}\Big|_{s=3}$

$$\mathcal{L}\{t^2 \sin t\} = (-1)^2 \frac{d^2}{ds^2} \mathcal{L}\{\sin t\} = \frac{d^2}{ds^2}\left(\frac{1}{s^2+1}\right) = \frac{d}{ds}\left(\frac{-2s}{(s^2+1)^2}\right)$$

$$= \frac{(s^2+1)^2(-2) - (-2s)(2)(s^2+1)(2s)}{(s^2+1)^4} = \frac{6s^2-2}{(s^2+1)^3}$$

so $\int_0^\infty t^2 e^{-3t} \sin t \, dt = \frac{6s^2-2}{(s^2+1)^3}\Big|_{s=3} = \frac{52}{1000} = \frac{13}{250}$

4-94

Sketch carefully the function $F(t) = t^2 - (t-2)^2 \mathcal{U}(t-2)$ for $t \geq 0$.

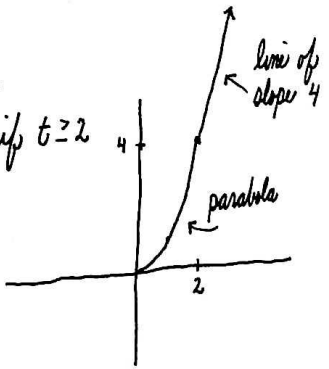
****

$$\mathcal{U}(t-2) = \begin{cases} 1 \text{ if } t \geq 2 \\ 0 \text{ if } t < 0 \end{cases}$$

So $F(t) = \begin{cases} t^2 \text{ if } t < 2 \\ t^2 - (t-2)^2 \text{ if } t \geq 2 \end{cases}$

$t^2 - (t-2)^2 = 4t - 4$

$F(t) = \begin{cases} t^2 \text{ if } t < 2 \\ 4t-4 \text{ if } t \geq 2 \end{cases}$



4-96

Find the Laplace transform of the solution of $ty'' + y' + ty = 0$ satisfying initial conditions $y(0) = 1,\ y'(0) = 0$. Do not invert.

****

Let $\bar{y} = \mathcal{L}\{y(t)\}$. The third term of the equation is a product and has transform $\mathcal{L}\{ty\} = -\frac{d\bar{y}}{ds}$. The first term is also a product and has transform $\mathcal{L}\{ty''\} =$

$-\frac{d}{ds}\mathcal{L}\{y''\} = -\frac{d}{ds}(s^2\bar{y} - s) =$

$-s^2\frac{d\bar{y}}{ds} - 2s\bar{y} + 1$. Hence the transform of the entire equation is

$$\left(-s^2\frac{d\bar{y}}{ds} - 2s\bar{y} + 1\right) + (s\bar{y} - 1) - \frac{d\bar{y}}{ds} = 0.$$

Simplifying algebraically, we get $(s^2+1)\frac{d\bar{y}}{ds} + s\bar{y} = 0$. This is a first-order D.E. in the variable $\bar{y}$ and is separable, in the form $\frac{d\bar{y}}{\bar{y}} = -\frac{s}{s^2+1}$.

Integrating, $\ln\bar{y} = -\frac{1}{2}\ln(s^2+1) + C$

$= \ln(s^2+1)^{-1/2} + C.$

Thus $\bar{y} = \dfrac{C}{\sqrt{s^2+1}}$.

$$= \frac{A(s+2)(s+1)(s^2+1) + B(s-2)(s+1)(s^2+1) + C(s-2)(s+2)(s^2+1) + (Ds+E)(s-2)(s+2)(s+1)}{(s-2)(s+2)(s+1)(s^2+1)}.$$

Equating the numerators, we obtain the equation (*)

$$s^3+2s+1 = A(s+2)(s+1)(s^2+1) + B(s-2)(s+1)(s^2+1) + C(s-2)(s+2)(s^2+1) + (Ds+E)(s-2)(s+2)(s+1)$$

We find A, B, C, D, E by plugging in (*) convenient values for s:

$s=2 \Rightarrow 2^3+2\cdot 2+1 = A(2+2)(2+1)(2^2+1) \Rightarrow A = \frac{13}{60}.$

$s=-2 \Rightarrow (-2)^3+2(-2)+1 = B(-2-2)(-2+1)(4+1) \Rightarrow B = -\frac{11}{20}.$

$s=-1 \Rightarrow (-1)^3+2(-1)+1 = C(-1-2)(-1+2)(1+1) \Rightarrow C = \frac{1}{3}.$

Finally, to find $Ds+E$, we put $s^2=-1$. Then the left hand side of (*) is $\underbrace{s^3}_{s^2\cdot s} + 2s + 1 = -s + 2s + 1 = \underbrace{s}_{i\sqrt{1}} + 1 = i\sqrt{1} + 1$

While the right hand side of (*) is

$$(\underbrace{Ds}_{Di\sqrt{1}} + E)(\underbrace{s^2}_{-1} - 4)(\underbrace{s}_{i\sqrt{1}} + 1) = (Di\sqrt{1} + E)(-5)(i\sqrt{1}+1)$$

By (*) we have

$$(i\sqrt{1}+1) = (Di\sqrt{1} + E)(-5)(i\sqrt{1}+1)$$

Hence

$$\underbrace{D}_{\text{real numbers}}i\sqrt{1} + \underbrace{E}_{\text{real numbers}} = -\frac{1}{5} \quad \Longrightarrow \quad D \equiv 0 \text{ and } E \equiv -\frac{1}{5}$$

In conclusion,

$$\underbrace{X(s)}_{\downarrow x(t)} = \frac{13}{60}\underbrace{\frac{1}{s-2}}_{\downarrow e^{2t}} - \frac{11}{20}\underbrace{\frac{1}{s+2}}_{\downarrow e^{-2t}} + \frac{1}{3}\underbrace{\frac{1}{s+1}}_{\downarrow e^{-t}} - \frac{1}{5}\underbrace{\frac{1}{s^2+1}}_{\downarrow \sin t}$$

The correct answer is (B).

Therefore $\frac{dy}{dx}$ for the orthogonal family is

equal to $-\left(\frac{y^2-x^2}{-2xy}\right) = \frac{y^2-x^2}{2xy}$

To obtain the integral form of the equation of the orthogonal family, we integrate $\frac{dy}{dx} = \frac{y^2-x^2}{2xy}$. This is a homogeneous equation.

Substitute $y = vx$, $\frac{dy}{dx} = v + x\frac{dv}{dx}$

$$v + x\frac{dv}{dx} = \frac{x^2(v^2-1)}{x^2(2v)} = \frac{v^2-1}{2v}$$

$$x\frac{dv}{dx} = \frac{v^2-1}{2v} - v = \frac{-1-v^2}{2v}$$

that is, $$\frac{2v}{1+v^2}dv + \frac{1}{x}dx = 0$$

Integrating $$\ln(1+v^2) + \ln|x| = C_2$$

$$\ln|x(1+v^2)| = C_2$$

$$x(1+v^2) = C_1 \qquad C_1 = e^{C_2} > 0$$

$$x\left(1+\frac{y^2}{x^2}\right) = C_1$$

$$x + \frac{y^2}{x} = C_1$$

that is $x^2 + y^2 - C_1x = 0$, where C_1 is a constant. Therefore the orthogonal family consists of circles passing through the origin and having centers on x-axis.

5
APPLICATIONS OF ORDINARY DIFFERENTIAL EQUATIONS

ORTHOGONAL TRAJECTORIES

5-1

Find the equation of the orthogonal trajectories of the family of curves given by $x^2 + y^2 - 2cy = 0$.

****

The given family consists of circles passing through the origin with their centers on y-axis.
Differentiating with respect to x

$$2x + 2y\frac{dy}{dx} - 2c\frac{dy}{dx} = 0$$

$$(y-c)\frac{dy}{dx} = -x, \text{ that is } \frac{dy}{dx} = -\frac{x}{y-c}$$

Substituting $c = \frac{x^2+y^2}{2y}$ from the equation of the given family

$$\frac{dy}{dx} = \frac{-x}{y - \frac{x^2+y^2}{2y}} = \frac{-2xy}{y^2-x^2}$$

5-3

Find the orthogonal trajectory of the family of lines of slope c and passing through the point (1,5).

****

The family of lines with slope c, through (1,5) is:

① $y-5 = c(x-1)$ or

$y = c(x-1)+5.$

$y' = c$ [using ① $c = \frac{y-5}{x-1}$]

so $y' = \frac{y-5}{x-1}$

$y'_{orthog} = -\frac{1}{\frac{y-5}{x-1}} = \frac{1-x}{y-5}$

$\Rightarrow \frac{dy}{dx} = \frac{1-x}{y-5}$. Separate variables

$(y-5)dy = (1-x)dx \Rightarrow \frac{y^2}{2} - 5y = x - \frac{x^2}{2} + C_1$

or $y^2 - 10y = 2x - x^2 + 2C_1$

$y^2 - 10y + x^2 - 2x = 2C_1$

$y^2 - 10y + 25 + x^2 - 2x + 1 = 2C + 25 + 1 = C_2$

or $(y-5)^2 + (x-1)^2 = C_2$, the family of circles with center at (1,5).

5-2

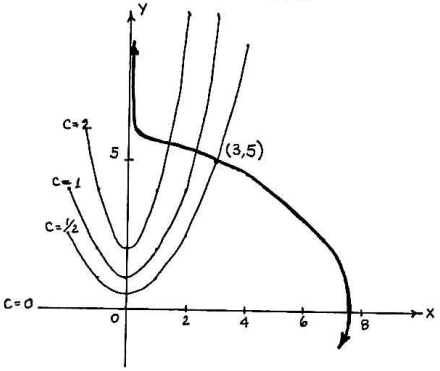
(a) Sketch the family of curves: $y = cx^2 + c$
For $c = -2, -1, -1/2, 0, 1/2, 1, 2$

(b) Find the orthogonal trajectories for the family in part (a).

(c) Sketch, on the graph for part (a), the orthogonal trajectory which passes through the point P (3,5).

****

a) Notice that the graph is symmetric in the x-axis, so we show only the portion for $y > 0$.



b) Now, $y = cx^2 + c \Rightarrow c = \frac{y}{x^2+1}$ and $y' = 2cx$

Thus $y' = \frac{2xy}{x^2+1}$, and the orthogonal trajectories are given by the differential equation

$$y' = \frac{-(x^2+1)}{2xy} \Rightarrow 2y\,dy = \frac{-(x^2+1)}{x}\,dx$$

So $y^2 = \frac{-x^2}{2} - \ln x + c$ or $\underline{\underline{2y^2 + x^2 + \ln x^2 = c}}$

c) Now, the point (3,5) lies on $y = \frac{1}{2}x^2 + \frac{1}{2}$ as indicated on the graph. The OT's do not cross the y-axis.

5-5

Find the orthogonal trajectories of the family of curves $y = cx^3$.

****

Step 1. Find the differential equation for the family of curves $y = cx^3$ (1)

Differentiate both side of (1) with respect to x

$$\frac{dy}{dx} = 3cx^2 \qquad (2)$$

Eliminate c from (2) by using (1)

$$\frac{dy}{dx} = \frac{3y}{x}$$

Step 2. Set up the differential equation for the orthogonal trajectories

$$\frac{dy}{dx} = -\frac{1}{\frac{3y}{x}} = -\frac{x}{3y} \qquad (3)$$

Step 3. Solve equation (3). This is an equation with seperable variables.

$$y\,dy = -\frac{1}{3}x\,dx$$

Integrate both side $\frac{1}{2}y^2 = -\frac{1}{6}x^2 + C$

Hence the family of orthogonal trajectories is

$$y^2 = c - \frac{1}{3}x^2$$

with any constant c .

5-4

Find the orthogonal trajectories for the family of curves given by the equation $y^3 = c(x^4 + 2)$.

****

$$\frac{d}{dx}(y^3) = \frac{d}{dx}\left(c(x^4+2)\right) \Rightarrow 3y^2\frac{dy}{dx} = c(4x^3)$$

WE ELIMINATE THE PARAMETER c USING $c = \frac{y^3}{x^4+2}$ AND OBTAIN

$$3y^2\frac{dy}{dx} = \left(\frac{y^3}{x^4+2}\right)(4x^3)$$

$$3y^2\frac{dy}{dx} = \frac{4x^3y^3}{x^4+2}$$

$$\therefore \quad \frac{dy}{dx} = \frac{4x^3y}{3(x^4+2)}$$

THE ORTHOGONAL TRAJECTORIES ARE FOUND BY CONSIDERING THE EQUATION

$$\frac{dy}{dx} = -\frac{3(x^4+2)}{4x^3y}$$

SINCE WE WANT THE TANGENT SLOPES TO BE THE NEGATIVE RECIPROCALS OF THOSE FOR THE ORIGINAL CURVES. THIS EQUATION IS SEPARABLE, SO

$$4y\,dy = -\frac{3(x^4+2)}{x^3}\,dx$$

$$\int 4y\,dy = \int(-3x - 6x^{-3})\,dx$$

$$2y^2 = -\frac{3}{2}x^2 - 6\left(\frac{x^{-2}}{-2}\right) + k$$

$$2y^2 = -\frac{3}{2}x^2 + \frac{3}{x^2} + k$$

OR: $$4x^2y^2 = -3x^4 + 6 + k_1x^2$$

WHERE $k_1 = 2k$

used

5-7

Consider a family of curves f(x,y) = k. Another family of curves, g(x,y) = k, is said to form a set of orthogonal trajectories of the first family if every intersection of a curve in the first family with a curve in the second family is at right angles.

(a) Find the orthogonal trajectories of $y = ce^{-2x}$

(b) Draw a sketch with one member of each family.

****

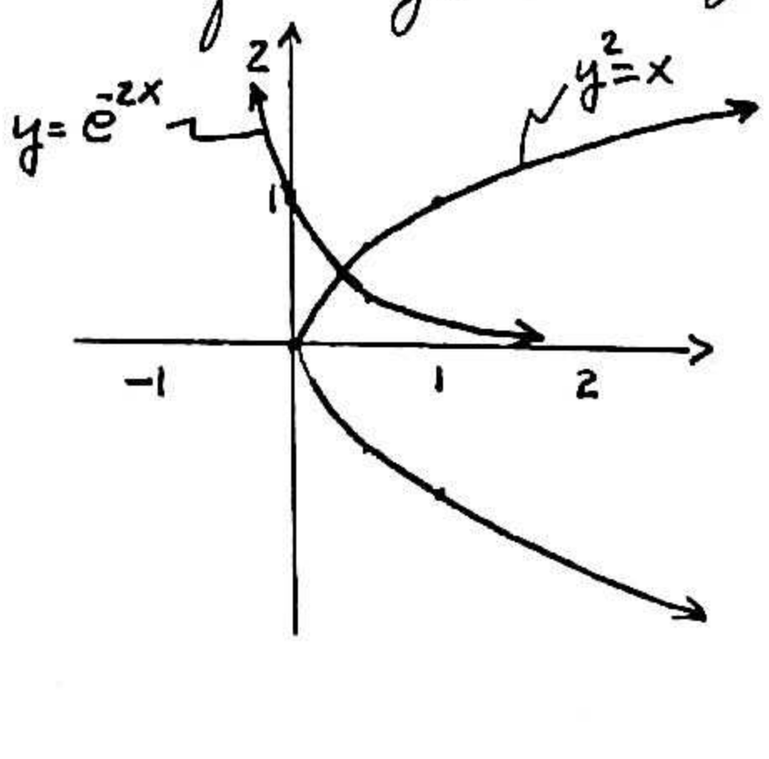
(a) Note that if $y = ce^{-2x}$, $\frac{dy}{dx} = -2ce^{-2x}$. Now, for every orthogonal trajectory we must have, if the intersections are to be at right angles, $\frac{dy}{dx} = -\frac{1}{(-2ce^{-2x})} = \frac{1}{2ce^{-2x}}$

or $\frac{dy}{dx} = \frac{e^{2x}}{2c}$ but $y = ce^{-2x} \Rightarrow \frac{y}{c} = e^{-2x} \Rightarrow \frac{c}{y} = e^{2x}$

$\therefore \frac{dy}{dx} = \frac{1}{2c} \cdot \frac{c}{y} = \frac{1}{2y} \Rightarrow y\,dy = \frac{1}{2}\,dx$

$\Rightarrow \frac{y^2}{2} = \frac{x}{2} + k_1 \Rightarrow \underline{\underline{y^2 = x + k}}$ $(k = 2k_1)$

(b.) Using $y = e^{-2x}$ (first family) and $y^2 = x$ (from the second family)



5-6

Given that the families of curves $y^3 = C_1 x$ and $x^2 + ay^2 = C_2$ are orthogonal, what is the value of the constant "a".

****

The family $y^3 = C_1 x$ has slope function determined as follows.

(implicit diff) $y^3 = C_1 x$ $\quad y' = \frac{C_1}{3y^2}$ $\qquad$ From $y^3 = C_1 x$, $C_1 = \frac{y^3}{x}$

$$y' = \frac{y^3}{x} \cdot \frac{1}{3y^2}$$

$$\frac{dy}{dx} = \frac{y}{3x}$$

The family $x^2 + ay^2 = C_2$ has slope function determined as follows.

$$x^2 + ay^2 = C_2$$

(implicit diff) $\quad 2x + 2ayy' = 0$

$$\frac{dy}{dx} = -\frac{x}{ay}$$

Orthogonality requires that the product of the slope functions equal -1.

Thus $-\frac{x}{ay} \cdot \frac{y}{3x} = -1$

$$a = \tfrac{1}{3}$$

5-9

Show that the family of ellipses $\frac{x^2}{4+k} + \frac{y^2}{1+k} = 1$, $k > -1$ satisfy the differential equation $p^2xy + p(x^2-y^2-3) - xy = 0$ where $p = \frac{dy}{dx}$ and deduce that the family of orthogonal trajectories to the above is the family of hyperbolas $\frac{x^2}{4+k} + \frac{y^2}{1+k} = 1$, $-4 < k < -1$.

****

Differentiate the above equation with respect to x,

$$\frac{2x}{4+k} + \frac{2y}{1+k}\frac{dy}{dx} = 0,$$

i.e. $\frac{2x}{4+k} + \frac{2py}{1+k} = 0$, i.e. $\frac{x}{4+k} + \frac{py}{1+k} = 0$

Solve for k, to obtain $k = -\frac{(x+4py)}{(x+py)}$.

Substitute in the original equation, to obtain

$$\frac{x^2}{\left[4 - \frac{(x+4py)}{(x+py)}\right]} + \frac{y^2}{\left[1 - \frac{(x+4py)}{(x+py)}\right]} = 1$$

i.e. $\frac{x^2(x+py)}{4(x+py)-(x+4py)} + \frac{y^2(x+py)}{(x+py)-(x+4py)} = 1$,

which simplifies to

$$\frac{x(x+py)}{3} + \frac{y(x+py)}{-3p} = 1,$$

which gives us

$$p^2xy + p(x^2-y^2-3) - xy = 0.$$

Replace p by $-\frac{1}{p}$ to obtain the D.E. of the orthogonal trajectories,

$$\frac{1}{p^2}xy - \frac{1}{p}(x^2-y^2-3) - xy = 0$$

i.e. $xy - p(x^2-y^2-3) - p^2xy = 0$

i.e. $p^2xy + p(x^2-y^2-3) - xy = 0$

This shows us that the orthogonal trajectories satisfy the same first order differential equation.

5-8

Find the orthogonal trajectories of the following family of curves

$$y = cx + 1.$$

Orthogonal trajectories are another family of curves which intersect the given family at right angles.

****

The family $y = cx + 1$ represents straight lines through $(0,1)$. First we find the governing differential equation of the given family. Differentiate w.r.t. x

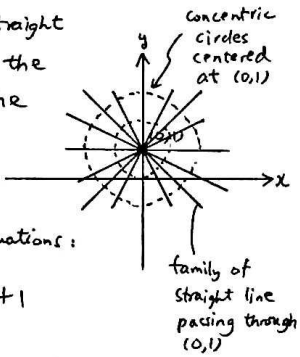
$$\frac{dy}{dx} = c$$

Eliminate c from the two equations:

$$\frac{dy}{dx} = c \quad \text{and} \quad y = cx + 1$$

we obtain

$$\frac{dy}{dx} = \frac{y-1}{x}.$$



The slope of any curve of the orthogonal family at (x,y) is given by

$$\frac{dy}{dx} = \frac{-1}{\frac{y-1}{x}} = \frac{-x}{y-1}$$

which is the governing differential equation for the orthogonal trajectories. Solving the differential equation by separation of variables

$$(y-1)\,dy + x\,dx = 0$$

$$\frac{y^2}{2} - y + \frac{x^2}{2} = A$$

$$(y-1)^2 + x^2 = (\sqrt{2A+1})^2 = r^2 \ (\text{say})$$

Hence the orthogonal trajectories are circles centered at $(0,1)$ with varying radius r. (See the diagram.)

5-10

Find the orthogonal trajectories associated with the following C-parameter family of curves, $f(x,y,C) = 0$:

$$\underline{2\cdot(x^2 + y) = 1 + C\cdot e^{-2y}} \ .$$

****

We first develop the c-eliminant ; that is, the first order differential equation absent in c whose solution is the given family of curves.

Implicit differentiation gives :

$$2\left(2x + \frac{dy}{dx}\right) = -2c\cdot e^{-2y}\cdot\frac{dy}{dx} \quad , \text{ or}$$

$$\frac{2x + \frac{dy}{dx}}{-\left(\frac{dy}{dx}\right)} = c\cdot e^{-2y} \ .$$

the elimination of c gives $\underline{2(x^2+y) = 1 + \left[\dfrac{2x + \frac{dy}{dx}}{-\left(\frac{dy}{dx}\right)}\right]}$.

Simplification yields the c-eliminant as $\underline{\dfrac{dy}{dx} = \dfrac{-x}{x^2+y}}$.

From the condition of perpendicularity, the relationship between the orthogonal trajectories (O.T.) and the given family of curves ($f=0$) is :

$$\left.\frac{dy}{dx}\right|_{O.T.} = \frac{-1}{\left(\frac{dy}{dx}\right)_f} \ .$$

Thus, the ordinary differential equation for the desired orthogonal trajectories is $\underline{\dfrac{dy}{dx} = \dfrac{x^2+y}{x}}$, or expressed as

$\underline{\dfrac{dy}{dx} + \left(\frac{-1}{x}\right)\cdot y = x}$. This equation is linear, first order, of

But $\frac{x^2}{4+k} + \frac{y^2}{1+k} = 1$, $k > -4$, $k \neq -1$ is the general solution of this differential equation.

(Note : $k < -4$ would make the left side negative)

Let $\frac{x^2}{4+c} + \frac{y^2}{1+c} = 1$, $c > -1$ be a member of the family of ellipses and consider any trajectory

$$\frac{x^2}{4+k} + \frac{y^2}{1+k} = 1 , \quad k > -4 , \quad k \neq -1 .$$

For the latter to be orthogonal to the former it is necessary that they intersect. Let the point of intersection be (x_0, y_0). Then

$$\frac{x_0^2}{4+c} + \frac{y_0^2}{1+c} - \frac{x_0^2}{4+k} - \frac{y_0^2}{1+k} = 0$$

i.e. $$(k-c)\left[\frac{x_0^2}{(4+c)(4+k)} + \frac{y_0^2}{(1+c)(1+k)}\right] = 0$$

If $k = c$, we have the same curve.

If $k > -1$, then the expression within brackets is positive and we must have $k = c$.

Therefore , $-4 < k < -1$ is the only range of values of k for which $\frac{x^2}{4+k} + \frac{y^2}{1+k} = 1$ is orthogonal to $\frac{x^2}{4+c} + \frac{y^2}{1+c} = 1$, $c > -1$. Thus, the family of orthogonal trajectories is $\frac{x^2}{4+k} + \frac{y^2}{1+k} = 1$, $-4 < k < -1$.

5-11

Find the family of orthogonal trajectories of the family of curves $x^3+y^3=a^3$ where a is an arbitrary constant.

****

$$x^3+y^3=a^3$$

$$3x^2+3y^2y'=0$$

$$y'=-\frac{x^2}{y^2}$$

For the orthogonal trajectory, $\frac{dy}{dx}=\frac{y^2}{x^2}$

$$y^{-2}dy=x^{-2}dx$$

$$-\frac{1}{y}=-\frac{1}{x}-C$$

$$\frac{1}{y}=\frac{1}{x}+C$$

$$x=y+Cxy$$

5-12

Find the orthogonal trajectories to the family of ellipses

$$x^2+\frac{y^2}{2}=c^2$$

****

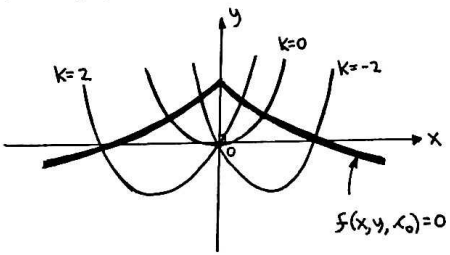
First take the derivative of the given equation, and eliminate the arbitrary constants:

the form $\frac{dy}{dx} + p(x)\cdot y = q(x)$. Its integrating factor is $\mu = e^{\int p(x)\,dx}$, and the differential equation admits the solution $y\cdot\mu = \int \mu\cdot q\cdot dx + K$. Herein, $\mu = e^{\int -\frac{1}{x}\,dx} = e^{-\ln x} \equiv \frac{1}{x}$.

Then, the solution is $y\cdot\frac{1}{x} = \int \frac{1}{x}\cdot x\cdot dx + K = x + K$, or

$$y = x\cdot(x+K).$$

The required orthogonal trajectories describe a K-parameter family of parabolas. Note, in the sketch below, that each member of the transcendental equation $f(x,y,c)=0$ cuts each member of the K-parameter family of parabolas at right angles.



Here N_0 represents the initial bacterial population and k is a constant.

The solution of equation (1) subject to the initial condition (2) is:

$$N(t) = N_0 e^{kt}. \qquad (3)$$

Since the population doubles in three hours, i.e. when $t = 3$, $N = 2N_0$, it follows from equation (3) that

$$2 = e^{3k}.$$

Therefore $$k = \frac{\ln 2}{3}.$$

Equation (3) now becomes

$$N(t) = N_0 e^{\frac{\ln 2}{3} t}. \qquad (4)$$

Using $N = 3N_0$ in this equation, we find that the time at which the population triples is given by

$$3 = e^{\frac{\ln 2}{3} t}.$$

Solving for t, we obtain

$$t = \frac{3 \ln 3}{\ln 2} \approx 4.75 \text{ hours}$$

$$2x + y\frac{dy}{dx} = 0 \quad \text{or} \quad \frac{dy}{dx} = -\frac{2x}{y}$$

The orthogonal trajectories will have negative reciprocal slope:

$$\frac{dy}{dx} = \frac{y}{2x} \Rightarrow 2\frac{dy}{y} = \frac{dx}{x} \Rightarrow$$

$$2\ln|y| = \ln|x| + C \quad \text{or}$$

$$\boxed{y^2 = Kx}.$$

GROWTH AND DECAY

used

5-13

In a certain culture the rate of bacterial growth is proportional to the number of bacteria present at any time. If the bacterial population doubles in three hours, how long will it take to triple?

* *

Denoting the number of bacteria present at any time t by $N(t)$, we have

$$\frac{dN}{dt} = kN \qquad (1)$$

with

$$N(0) = N_0. \qquad (2)$$

5-15

Suppose at the beginning there are 10^6 bacteria in a culture plate and the rate of growth of the bacteria is proportional to the number of the bacteria present. Find the number of bacteria at any time t. (Assume that the proportional constant is k.)

****

Assume that the number of the bacteria at time t is $y(t)$, then the rate of growth is $\frac{dy}{dt}$.

And by the assumption of the problem

$$\frac{dy}{dt} = ky$$

This is a separable equation. Separate the variables

$$\frac{dy}{y} = k\,dt$$

and integrate $\ln y = kt + C'$

i.e. $$y = Ce^{kt} \qquad (1)$$

In order to find constant C, we apply the initial condition $y(0) = 10^6$. Let $t=0$ in (1), we get $C = 10^6$. Therefore at time t, there are $10^6 e^{kt}$ bacteria in the culture plate.

5-14

Consider an investment plan in which interest is compounded continuously and additions to the account are deposited continuously. The rate of change of the balance of the account is equal to the net rate that funds are added to the account, in this case due to interest accumulation and continuous deposits. Suppose that I open an IRA account at age 27 with an initial deposit of $2000 and make deposits in a continuous manner of $1500 per year thereafter. Assuming a constant interest rate of 7%, what will be the balance of the IRA when I am 65?

****

$\frac{dA}{dt} = rA + k$ where A = account balance, r = interest rate, k = annual deposit rate

$\frac{dA}{dt} - rA = k$ First order linear $\mu(t) = e^{\int -r\,dt} = e^{-rt}$

$e^{-rt}\frac{dA}{dt} - re^{-rt}A = ke^{-rt}$

$\frac{d}{dt}\left(e^{-rt}A\right) = ke^{-rt}$

$e^{-rt}A = \frac{-k}{r}e^{-rt} + C$

$A(t) = \frac{-k}{r} + Ce^{rt}$

$r = .07$, $k = 1500$: $A(t) = \frac{-1500}{.07} + Ce^{.07t}$

$A(0) = 2000$: $2000 = \frac{-1500}{.07} + C$ $\quad C = 2000 + \frac{1500}{.07}$

so $A(t) = \frac{-1500}{.07} + \left(2000 + \frac{1500}{.07}\right)e^{.07t}$

$A(38) = 313,513.06$

[note that total investments were $59,000]

5-17

The rate of decay of a radioactive substance is proportional to the amount present.

How long will it take for half of the substance to decay if it takes 4 years for a quarter of the substance to decay?

****

Let the amount present be $x(t)$ at time t.

Then $\frac{dx}{dt} = -kx$, where $k>0$ is the constant of proportionality. This is a linear DE.

Integrating factor $\mu(t) = e^{\int k\,dt} = e^{kt}$

$$\Rightarrow e^{kt}\left(\frac{dx}{dt} + kx\right) = 0 \Rightarrow \frac{d}{dt}\left(xe^{kt}\right) = 0$$

Integrating, $xe^{kt} = x_0 \Rightarrow x(t) = x_0 e^{-kt}$,

where x_0 is the amount present at $t=0$.

The other information is $x(4) = \frac{3}{4}x_0 = x_0 e^{-4k}$

Thus $e^{-4k} = \frac{3}{4} \Rightarrow -4k = \ln\left(\frac{3}{4}\right)$, $k = \underline{\frac{1}{4}\ln\left(\frac{4}{3}\right)}$.

To find t_1 such that $x(t_1) = \frac{1}{2}x_0$,

$\frac{1}{2}x_0 = x_0 e^{-kt_1} \Rightarrow e^{-kt_1} = 2^{-1} \Rightarrow \underline{kt_1 = \ln 2}$.

$$\boxed{t_1 = \frac{\ln 2}{\frac{1}{4}\ln\left(\frac{4}{3}\right)} \approx 9.638 \text{ years}}$$

5-16

A population of organisms governed by the law of simple population growth has a birth rate of 0.35 per member per week, that is ,

$$\frac{dp}{dt} = 0.35\,p$$

How long does it take for the population to triple?

****

Let P_0 be the initial population, that is, $P(0)=P_0$. We solve the differential equation

$$\frac{dp}{dt} = 0.35\,p$$

with the initial condition: $P(0) = P_0$. Separation of variables gives

$$\frac{dp}{p} = 0.35\,dt$$

and upon integration

$$\int_{P_0}^{P} \frac{dp}{p} = \int_0^t 0.35\,dt$$

$$\log_e \frac{P}{P_0} = 0.35t$$

giving

$$P = P_0 e^{0.35t}$$

We try to find t such that $P(t) = 3P_0$, that is,

$$P_0 e^{0.35t} = 3P_0.$$

Taking logarithm on both sides

$$0.35t = \ln 3$$

or $$t = \frac{\ln 3}{0.35}.$$

5-19

A radioactive isotope decays so as to satisfy the differential equation:

$\frac{dQ}{dt} = -.05Q$, where Q is measured in milligrams and t in years.

(a) Determine the half-life, τ, of this isotope.

(b) If the initial amount of the isotope was 10 milligrams, how much will remain after $t = 10\ln(10)$ years?

****

(a) If $\frac{dQ}{dt} = -.05Q \Rightarrow \int \frac{dQ}{Q} = \int -.05\,dt$

$\Rightarrow \ln Q = -.05t + C$

exponentiating: $\Rightarrow Q(t) = e^{-.05t+C} = e^{-.05t} \cdot e^{C}$

$\therefore Q(t) = Ae^{-.05t}$ ($e^C = A$)

where $Q(t)$ measures the milligrams left after t years.

Note $Q(0) = Ae^0 = A$, since A is the original amount of the isotope. If τ is the half-life time - then $\frac{A}{2} = Q(\tau) = Ae^{-.05\tau}$, dividing out by A

$\Rightarrow \frac{1}{2} = e^{-.05\tau} \Rightarrow 2^{-1} = e^{-.05\tau} \Rightarrow \ln 2^{-1} = \ln e^{-.05\tau}$

$\Rightarrow -\ln 2 = -.05\tau$

$\therefore \tau = \frac{\ln 2}{.05}$ yrs. [$\tau \cong 13.862944$ yrs]

(b) From (a) we know that $Q(t) = Ae^{-.05t}$ and A = initial amount ($t=0$)

Here A = 10 milligrams

$\therefore Q(t) = 10e^{-.05t}$ (t in yrs, Q in milligrams)

$\therefore Q(t = 10\ln 10) = 10e^{-.05(10\ln 10)}$

$= 10e^{-.5\ln 10} = 10e^{\ln 10^{-\frac{1}{2}}} = 10 \cdot 10^{-\frac{1}{2}}$

$Q(t = 10\ln 10) = 10^{\frac{1}{2}} \cong \sqrt{10}$ milligrams.

5-18

A population of size P has a growth rate proportional to P. If the population doubles in 3 years, in how many years, to the nearest tenth, will the population be 7 times what it was initially? Let the initial population be P_0 at time t = 0.

****

$\frac{dP}{dt} = kP$ for some constant k.

$\frac{dP}{P} = k\,dt$

$\ln P - \ln C = kt$

$\ln\left(\frac{P}{C}\right) = kt$

$P = Ce^{kt}$

When $t=0$, $P=P_0$

So $P_0 = Ce^0$

$P_0 = C$

So $P = P_0 e^{kt}$

When $t=3$, $P=2P_0$

$2P_0 = P_0 e^{3k}$

$2 = e^{3k}$

$\ln 2 = 3k$

$.231 = k$

$P = P_0 e^{.231t}$

Let $P = 7P_0$

$7 = e^{.231t}$

$\ln 7 = .231t$

$\frac{\ln 7}{.231} = t$

8.4 years $= t$

5-21

Bacteria is growing in a dish, such that the rate of change in bacteria is proportional to the number present. The dish can hold 1000 cc of bacteria. Each one takes up $\frac{1}{4}$ cc. At the start there are 20 bacteria or 5 cc of bacteria and two hours later there are 50 cc or 200 bacteria present. Use differential equations to find the time when the dish will be full.

****

Let $N(t)$ be the number of bacteria present at time t. Then $N(0)=20$, $N(2)=200$, and $\frac{dN}{dt}=kN$. Thus the problem in mathematical terms is to solve $\frac{dN}{dt}=kN$ given $N(0)=20$, and $N(2)=200$. Finally, find t when $N(t)=4000$. The solution to the differential equation is $N(t)=20e^{kt}$. Since $N(2)=200$, $200=20e^{2k}$. Thus $10=e^{2k}$, so $\ln 10=2k$ and $k=\frac{1}{2}\ln 10$. Therefore to solve the problem we need to solve $4000=20e^{(\frac{1}{2}\ln 10)t}$, for t. Solving, it is found that

$$t=\frac{2\ln 200}{\ln 10}=4.602 \text{ hours.}$$

Thus approximately 4 hrs and 36 minutes after the start the dish will be full.

5-20

A rare coin purchased in 1960 for $6.00 was estimated to be worth $30.00 in 1980. If the value continues to increase at a rate proportional to its size, during what year will the value reach $300.00?

****

THE EXPONENTIAL GROWTH EQUATION $\frac{dx}{dt} = kx$ WITH $x(0) = x_0$ HAS THE GENERAL SOLUTION $x = x_0 e^{kt}$.

ASSIGNING $t = 0$ TO THE YEAR 1960, WE HAVE $x_0 = \$6.00$, AND WHEN $t = 20$ (IN 1980) WE HAVE $x(20) = \$30.00$. THUS

$$30.00 = 6.00\, e^{k \cdot 20}$$
$$5 = e^{20k}$$
$$\ln 5 = 20k$$
$$\tfrac{1}{20} \ln 5 = k$$

OUR EQUATION FOR THE VALUE x AT TIME t IS KNOWN:

$$x = x_0 e^{kt} = 6.00\, e^{(\frac{1}{20} \ln 5)t} = 6.00 \left(e^{\ln 5}\right)^{t/20}$$
$$x = 6.00\,(5)^{t/20}$$

TO DETERMINE THE YEAR IN WHICH THE VALUE WILL REACH $300.00, WE SET $x = 300$ AND SOLVE FOR t.

$$300 = 6\,(5)^{t/20}$$
$$50 = \frac{300}{6} = (5)^{t/20}$$
$$\ln 50 = \ln (5)^{t/20} = \tfrac{t}{20} \ln 5$$
$$\Longrightarrow \quad t = \frac{20 \ln 50}{\ln 5} \cong 48.6 \text{ YEARS}$$

THE COIN WILL REACH A VALUE OF $300 DURING THE YEAR 2008.

5-23

In the year 1900, the world population was one billion. Since then, the population $p(t)$ (measured in billions) has been growing at the rate $dp/dt = kp(10-p)$, where k is a constant of proportionality and where 10 represents the projected maximum possible population of 10 billion. Solve this equation for $p(t)$, where t denotes the number of years since 1900. Use $k = 0.00253$ to predict the population in 2000 and 2050.

****

Separating variables, $\frac{dp}{p(10-p)} = k\,dt$.

The left-side integrand can be expanded in partial fractions as follows:

$\frac{1}{p(10-p)} = \frac{1}{10}\left(\frac{1}{p} + \frac{1}{10-p}\right)$. Hence the equation integrates on both sides to

$\frac{1}{10}\left[\ln p - \ln(10-p)\right] = kt + C.$

Simplifying, $\ln\left(\frac{p}{10-p}\right) = 10kt + 10C.$

When $t = 0$ (the year 1900) $p = 1$, so we solve to get $10C = \ln(1/9)$. Exponentiating both sides, we get

$$\frac{p}{10-p} = e^{10kt + \ln(1/9)} = \frac{e^{10kt}}{9}.$$

5-22

An amount Q_0 of pure uranium is prepared at time $t = 0$ and allowed to decay; the differential equation governing its decay is

$$\frac{dQ_U}{dt} = -k_U Q_U$$

where Q_U is the amount of uranium, and k_U is its decay rate, a constant.

a) Write down Q_U as a function of time.

b) Uranium decays into (radioactive) thorium, the production and decay of which is governed by the equation

$$\frac{dQ_T}{dt} = -k_T Q_T - \frac{dQ_U}{dt}$$

where Q_T is the amount of thorium, and k_T is the thorium decay rate (Note: $k_T \neq k_U$), a constant. Find Q_T.

****

$\frac{dQ_U}{dt} = -k_U Q_U$ gives $Q_U = c e^{-k_U t}$. The initial value at $t=0$ is Q_0, so $c = Q_0$ and $Q_U = Q_0 e^{-k_U t}$.

$\frac{dQ_U}{dt} = -k_U Q_0 e^{-k_U t}$, so $\frac{dQ_T}{dt} + k_T Q_T = +k_U Q_0 e^{-k_U t}$.

The integrating factor here is $e^{k_T t}$, giving

$$\frac{d}{dt}\left(e^{k_T t} Q_T\right) = e^{k_T t}\frac{dQ_T}{dt} + k_T e^{k_T t} Q_T = k_U Q_0 e^{(k_T - k_U)t}$$

and $$e^{k_T t} Q_T = \int k_U Q_0 e^{(k_T - k_U)t} dt = \frac{k_U Q_0 e^{(k_T - k_U)t}}{k_T - k_U} + c$$

The initial value at $t=0$ is 0 as the sample is pure uranium then: $0 = \frac{k_U Q_0}{k_T - k_U} + c$, $c = \frac{-k_U Q_0}{k_T - k_U}$, so

$$e^{k_T t} Q_T = \frac{k_U Q_0 e^{(k_T - k_U)t} - k_U Q_0}{k_T - k_U}, \text{ and } Q_T = \frac{k_U Q_0}{k_T - k_U}\left(e^{-k_U t} - e^{-k_T t}\right).$$

5-24

John and Jane are comparing notes on their investment activity. John deposits $1,000 in Bank I , for a 10 year period, with his investment compounded continuously at 6% interest per annum.

Jane deposits $1,000 in Bank II, for a 10 year period, with her investment compounded once every 5 years (that is, the 'simple interest' principle is applied every five years).

Assuming that no 'interference' exists - that is, assume that no additional deposits and no withdrawals are made in the investments over the ten years of activity, determine the interest rate that Bank II must pay Jane if the yields are the same for the two investments at the end of ten years.

A useful number here is $e^{0.3} \approx 1.35$.

****

John's investment is compounded continuously ; that is

$\frac{dS}{dt} = r \cdot S$, where S = the amount of savings present at time t , and r = the annual rate of interest.

Via 'separation,' $\int \frac{dS}{S} = r \cdot \int dt$, yielding

$\ln S - \ln c \equiv \ln\left(\frac{S}{c}\right) = rt$, or $\underline{S = c \cdot e^{rt}}$.

At $t=0$, $S = S_0 = \$1{,}000$. Hence $c = 1{,}000$.

After 10 years, with $r = 6\%$, $S_{10} = S(10) = 1{,}000. \cdot e^{0.6}$

$= \$1{,}822.10$.

Jane's investment is compounded in a step-wise manner every 5 years. Here $\hat{S}_0 = \$1{,}000$.

$\hat{S}_5 = \hat{S}_0 + I \equiv \hat{S}_0 + \hat{S}_0 \cdot \hat{r} \cdot t = 1{,}000. + 1{,}000. \cdot \hat{r} \cdot 5$

$= 1{,}000. \cdot (1 + 5\hat{r})$;

$\hat{S}_{10} = \hat{S}_5 + \hat{S}_5 \cdot \hat{r} \cdot t \Big|_{t=5} = 1{,}000. \cdot (1+5\hat{r})^2$, with $\hat{r}$ = annual rate of interest.

Taking the reciprocal of both sides, $\frac{10-p}{p} = 9e^{-10Kt}$. Solving for p, we obtain $p(t) = \frac{10}{1+9e^{-10Kt}}$.

This function has an s-shaped graph called the logistic curve. Note that $p(0) = 1$ (as required) and that $\lim_{t\to\infty} p(t) = 10$, the maximum population (in billions of people). Using the value $K = 0.00253$, we calculate that $p(100) \cong 5.8$ (billion) and that $p(150) \cong 8.3$ (billion). These are our predicted populations in the years 2000 and 2050, respectively.

It is of interest to note that the inflection point of the curve occurred in the year 1987, when the population was five billion.

5-25

A bacterial population is known to have a rate of growth which is proportional to the size of the population. Between noon and 2 p.m., the size of the population triples. Given this and with no controls exerted, at what time will the population be one hundred times the size it was at noon?

****

LET B = POPULATION SIZE AT ANY TIME t IN HOURS, WITH B_0 REPRESENTING THE INITIAL POPULATION AT NOON $(t=0)$. WITH THIS, IF THE POPULATION TRIPLES BY 2 P.M., THEN WHEN $t=2$, $B=3B_0$.

$\frac{dB}{dt}$ REPRESENTS THE RATE OF CHANGE OF B WITH RESPECT TO t, AND IT IS GIVEN THAT $\frac{dB}{dt}$ IS PROPORTIONAL TO B.

$\therefore \frac{dB}{dt} = kB$, $\frac{dB}{B} = k\,dt$, AND BY INTEGRATING BOTH SIDES,

$\ln B = kt + C$. WITH $t=0$, $B=B_0$, $\ln B_0 = k(0) + C$, SO

$\ln B = kt + \ln B_0$, OR $\ln B - \ln B_0 = kt$, OR $\ln \frac{B}{B_0} = kt$,

SINCE $\ln P - \ln Q = \ln \frac{P}{Q}$ IS ONE OF THE LAWS OF LOGARITHMS.

WITH $t=2$, $B=3B_0$, $\ln \frac{3B_0}{B_0} = k(2)$ OR $k = \frac{1}{2}\ln 3$.

$\therefore \ln \frac{B}{B_0} = (\frac{1}{2}\ln 3)t$.

WE WANT TO FIND t WHEN $B = 100\,B_0$ (i.e., "ONE HUNDRED TIMES" THE SIZE AT NOON.)

$\therefore \ln \frac{100\,B_0}{B_0} = (\frac{1}{2}\ln 3)t$ OR $t = 2\,\frac{\ln 100}{\ln 3} = 8.38$ HOURS

THEN 8 HOURS, 23 MINUTES AFTER NOON, OR 8:23 P.M. IS OUR ANSWER.

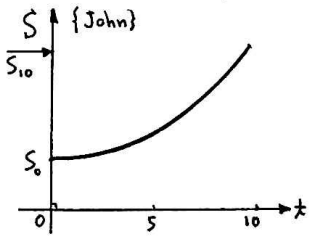
To determine the value of $\hat{r}$, we set the ten year yields equal to each other.

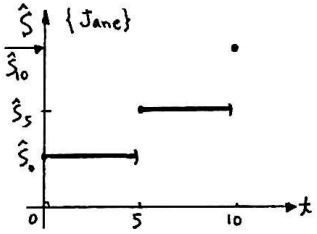
Thus $\hat{S}_{10}\Big|_{Jane} \equiv S_{10}\Big|_{John}$ implies

$$1{,}000 \cdot (1+5\hat{r})^2 = 1{,}000 \cdot e^{0.6}.$$

Solving for $\hat{r}$ gives $\hat{r} = \dfrac{e^{0.3}-1}{5} \approx \dfrac{1.35-1.00}{5} = 0.07 = 7\%$.

Note, per graphics below, the exponential growth vs step-wise linear growth.





this would give

$$0 < P(0) \leq P(0) - K \text{ , not true}$$

since $k > 0$. If $P(0) < K$ we have $0 > P(0) \geq P(0) - K$.

Thus unless $P(0) < 0$ (an unlikely event for a population size)

as $t \to \infty$, $P \to K$

MIXING PROBLEMS

5-27

A tank contains 100 gallons of pure water. Brine containing 2 pounds of salt per gallon is added to the tank at the rate of 5 gallons per minute.

(a) How much salt is present after 20 minutes if the well stirred mixture runs out at the rate of 6 gallons per minute?

(b) Find the maximum number of pounds of salt the tank will contain and the number of minutes needed to reach the maximum value.

****

5-26

A model that is often used to describe the growth of a population is the logistics equation:

$$\frac{dP}{dt} = c\,P\,(k-P),$$

where c and k are positive constants. Solve this for P. What can you say about solutions as t gets large?

****

The equation is separable: $\frac{dP}{P(k-P)} = c\,dt$.

Partial fractions gives $\left(\frac{1/k}{P} + \frac{-1/k}{P-k}\right)dP = c\,dt$ or

$\left(\frac{1}{P} - \frac{1}{P-k}\right)dP = ck\,dt$. Integrate:

$\ln\left|\frac{P}{P-k}\right| = ckt + a$. Exponentiate:

$\frac{P}{P-k} = Ae^{kct}$ or $\boxed{P = \frac{Ake^{kct}}{Ae^{kct}-1}}$

Since k and c are positive, as $t \to \infty$ $P \to k$. The only possible problem is if $Ae^{kct} - 1 = 0$ for some $t \geq 0$. Since k and c are positive, this happens only when $0 < A \leq 1$. Since $A = \frac{P(0)}{P(0)-k}$, this will happen only when $0 < \frac{P(0)}{P(0)-k} \leq 1$. For $P(0) > k$

$$\Rightarrow \frac{1}{6} = \left(\frac{100-t}{100}\right)^5 \Rightarrow \left(\frac{1}{6}\right)^{\frac{1}{5}} = \frac{100-t}{100}$$

$$.69882712 = \frac{100-t}{100} \Leftarrow \quad 100 - t = 69.882712$$

$\Rightarrow t = 30.11729$ Minutes to maximum

$\Rightarrow$ Maximum $= y(30.11729) = 116.47$ (lbs)

5-28

A beer manufacturer needs to convert a 40 gallon tank of regular beer into light beer by adding water. At time t=0, water is pumped into the tank at the rate of 2 gallons per minute, and the well stirred mixture leaves the tank at the same rate. The light beer has one third the calories of the regular beer (that is, it is two thirds water). How long will it take to convert the regular beer into light beer?

****

$X(t)$ = number of gallons of water in tank after t minutes

Then $\frac{dx}{dt}$ = rate in - rate out = $2 - 2\frac{x}{40}$, $X(0) = 0$.

The equation $\frac{dx}{dt} + \frac{1}{20}x = 2$ is linear, with an integrating factor of $e^{\int 1/20} = e^{\frac{1}{20}t}$. Then $(e^{\frac{1}{20}t}x)' = 2e^{\frac{1}{20}t}$,

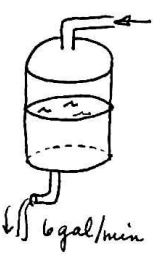
$e^{\frac{1}{20}t}x = \frac{2e^{\frac{1}{20}t}}{1/20} = 40e^{\frac{1}{20}t} + C$; $X(t) = 40 + Ce^{-\frac{1}{20}t}$

With t=0 and x=0, $0 = 40 + C$, so $C = -40$.

$X(t) = 40 - 40e^{-\frac{1}{20}t} = 40(1 - e^{-\frac{1}{20}t})$. The tank is two thirds water when $x = \frac{2}{3}40 = \frac{80}{3}$.

$\frac{80}{3} = 40(1 - e^{-\frac{1}{20}t})$, $\frac{2}{3} = 1 - e^{-\frac{1}{20}t}$, $e^{-\frac{1}{20}t} = \frac{1}{3}$,

so $t = -\frac{\ln 1/3}{1/20} = 20 \ln 3 \approx 22$ minutes.



Let $y(t)$ = amount (lbs) of salt in the tank after t minutes.

Note: $y(0) = 0$

$$\therefore \left(\frac{\text{lbs}}{\text{min}}\right)\frac{dy}{dt} = \text{rate in} - \text{rate out}$$

$$= 5\frac{\text{gal}}{\text{min}} \cdot 2\frac{\text{lbs}}{\text{gal}} - \frac{y(t)(\text{lbs})}{(100-t)\,\text{gal}} \cdot 6\frac{\text{gal}}{\text{min}}$$

$$\frac{dy}{dt} = 10\left(\frac{\text{lbs}}{\text{min}}\right) - \frac{6y}{(100-t)}\left(\frac{\text{lbs}}{\text{min}}\right)$$

$$\therefore \frac{dy}{dt} + \frac{6}{100-t}y = 10 \;;\; \mu = e^{\int \frac{6}{100-t}dt} = e^{-6\ln(100-t)} = e^{\ln(100-t)^{-6}}$$

$$\mu = (100-t)^{-6} \quad \text{(integrating factor)}$$

$$\therefore (100-t)^{-6}\left[\frac{dy}{dt} + \frac{6}{100-t}y\right] = 10(100-t)^{-6}$$

$$\Rightarrow \frac{d}{dt}\left[(100-t)^{-6}\cdot y\right] = 10(100-t)^{-6}$$

$$\Rightarrow (100-t)^{-6}\cdot y = \frac{10(100-t)^{-5}}{5} + C_1$$

$$\Rightarrow y(t) = 2(100-t) + C_1(100-t)^6$$

But $0 = y(0) = 2(100) + C_1(100)^6$

$0 = 200 + C_1(100)^6$

$\Rightarrow C_1 = -200(100)^{-6}$

$\Rightarrow y = 2(100-t) - 200(100)^{-6}(100-t)^6$

or $y = 2(100-t) - 200\left(\frac{100-t}{100}\right)^6$

(a) $y(t=20) = 2(80) - 200\left(\frac{80}{100}\right)^6 = 107.5712$ (lbs)

(b) To find the maximum lbs of salt, we need to examine $y'(t)$

$$y'(t) = -2 + 200(100)^{-6}\cdot 6(100-t)^5 \leq 0$$

$$\Rightarrow 2 = 200(100)^{-6}\cdot 6(100-t)^5 \Rightarrow 1 = 100^{-5}\cdot 6\cdot(100-t)^5$$

$$\Rightarrow 1 = 6\left(\frac{100-t}{100}\right)^5$$

5-30

Initially, 5 pounds of salt is in 10 gallons of water. A solution with 3 pounds of salt per gallon flows into the tank at a rate of 2 gallons per minute. A well mixed solution leaves the tank at the same rate.

a) Find the amount of salt in the tank at any time $t > 0$.

b) How much salt is in the tank after a "long time" ? (ie. what is the limiting amount of salt?)

****

a) Let $A(t)$ be the amount of salt at time t

$\frac{dA}{dt}$, the rate of change of A =

rate of salt entering - rate leaving

$$\frac{dA}{dt} = \frac{3\,lb}{gal} \cdot \frac{2\,gal}{min} - \frac{A}{10\,gal} \cdot \frac{2\,gal}{min}$$

$$\frac{dA}{dt} = 6 - \frac{1}{5}A \quad , \quad A(0) = 5$$

$\frac{dA}{dt} + \frac{1}{5}A = 6$ use integrating factor method $\mu = e^{\int \frac{1}{5}dt} = e^{\frac{1}{5}t}$

Solution: $Ae^{1/5\,t} = \int 6e^{1/5\,t}\,dt$

$$Ae^{1/5\,t} = 30e^{1/5\,t} + C$$

$$A = 30 + Ce^{-1/5\,t}$$

$A(0) = 5 = 30 + Ce^{-1/5\,0} \Rightarrow C = -25$

$$\underline{\underline{A = 30 - 25e^{-1/5\,t}}}$$

b) $\lim_{t \to \infty} (30 - 25 - e^{-1/5\,t}) = \underline{\underline{30\,lbs}}$

5-29

A 10% acid solution flows at a rate of two gallons per minute into a one hundred gallon tank which is initially full of fresh water. If the thoroughly stirred solution overflows at the same rate, in how many minutes will the solution in the tank be 5% acid?

****

LET W REPRESENT GALS. PURE ACID IN TANK AT ANY TIME t IN MIN.

WE KNOW THAT WHEN $t=0$, $W=0$ (INITIALLY FRESH WATER) AND THAT 10 % OF 2 OR 0.2 GALS. OF ACID ENTER PER MIN.

IN A 100 GAL TANK, $\frac{W}{100}$ IS THE AMOUNT OF ACID IN EACH GALLON OF TANK FLUID AT ANY TIME t AND SINCE 2 GALS./MIN OF MIXED SOLUTION FLOW OUT, $2\left(\frac{W}{100}\right) = \frac{W}{50}$ IS THE AMOUNT OF PURE ACID FLOWING OUT PER MIN.

THE RATE OF CHANGE OF W PER UNIT TIME IS INPUT - OUTPUT OR

$$\frac{dW}{dt} = 0.2 - \frac{W}{50} \quad \text{OR} \quad \frac{dW}{0.2 - \frac{W}{50}} = dt \quad \text{OR} \quad \frac{50\,dW}{10-W} = dt.$$

INTEGRATING BOTH SIDES, $-50 \ln (10-W) = t + C$. NOTE THAT $W \leq 10$, AND THAT AS t BECOMES VERY LARGE, W APPROACHES 10.

WHEN $t=0$, $W=0$ YIELDS $-50 \ln (10-0) = C$.

$\therefore -50 \ln (10-W) = t - 50 \ln 10$. THEN $t = 50 \ln 10 - 50 \ln (10-W)$

OR BY FACTORING AND LOGARITHM LAWS, $t = 50 \ln \frac{10}{10-W}$.

A 5% SOLUTION IMPLIES THAT $W = 5\%$ OF 100 GALS $= 5$ GALS.

$\therefore t = 50 \ln \frac{10}{10-5} = 50 \ln 2 = 50\,(0.693) = 34.7$ MINUTES.

5-32

Initially 50 pounds of salt is dissolved in a large tank holding 300 gallons of water. A brine solution is pumped into the tank at a rate of 3 gallons per minute and the well-stirred solution is then pumped out at the same rate. If the concentration of the solution entering is 2 pounds per gallon, determine the amount of salt in the tank at any time. How much salt is present (i) after 50 minutes (ii) after a long time?

****

Let $A(t)$ be the amount of salt (in pounds) present in the tank at any time. In each minute,

amount of salt in $= 2 \cdot 3 = 6$

amount of salt out $= \frac{A}{300} \cdot 3 = \frac{A}{100}$.

Therefore the rate of change of $A(t)$

$$\frac{dA}{dt} = 6 - \frac{A}{100},$$

with the initial condition: $A(0) = 50$.

Multiply the equation by the integrating factor $e^{t/100}$

$$\frac{d}{dt}\left[e^{t/100} A(t)\right] = 6e^{t/100}$$

$$A(t) = 600 + Ce^{-t/100}$$

with an arbitrary constant C. Using the initial condition we have $50 = 600 + C$ which gives $C = -550$.

The solution is $A(t) = 600 - 550e^{-t/100}$.

(i) $A(50) = 600 - 550e^{-1/2} \approx 266.41$ (ii) $\lim_{t \to \infty} A(t) = 600$.

5-31

A nitric acid solution flows at a constant rate of 4 L/min into a large tank that initially held 400 L of a 0.5% nitric acid solution. The solution inside the tank is kept well stirred and flows out of the tank at a rate of 8 L/min. If the solution entering the tank is 20% nitric acid, determine the amount of nitric acid in the tank after t minutes. When will the percentage of nitric acid in the tank reach 10%?

****

Let $A(t)$ be the amount of nitric acid in the tank at time t. NOTE: $A(0) = 400 \times (.005) = 2$.

The rate of acid entering $= (4 \text{ L/min})(.20) = .8 \text{ L/min}$

Now, at time t, the volume of solution in the tank is $400-4t$.

So, the rate of acid leaving $= \dfrac{8A(t)}{(400-4t)} = \dfrac{2A(t)}{(100-t)}$

Thus, $\dfrac{dA}{dt} = .8 - \dfrac{2A}{100-t}$

This equation is linear and $\mu(t) = e^{\int \frac{2}{100-t}\,dt} = (100-t)^{-2}$

So, $\dfrac{d}{dt}\left[(100-t)^{-2} \cdot A\right] = (.8)(100-t)^{-2}$

Which implies that $A = (.8)(100-t) + (100-t)^2 \cdot c$

Using $A(0) = 2$ gives $c = -78 \times 10^{-4}$

Thus $A(t) = (.8)(100-t) - (100-t)^2 (7.8 \times 10^{-3})$

We want $A(t)/(400-4t) = 0.1$

So $(0.1)(400-4t) = (.8)(100-t) - (100-t)^2 (7.8 \times 10^{-3})$

Which implies that $t = 48.7$ min.

PHYSICAL PROBLEMS

5-34

A free body of mass m is falling under gravity and is also subjected to a resistive force of magnitude KV^3, where V is the instantaneous velocity. Find the limiting velocity. (Hint: gravitational force = mg.)

****

From Newton's second law of motion

$$m\frac{dv}{dt} = mg - kv^3$$

where $\frac{dv}{dt}$ gives the acceleration and $mg - kv^3$ is the sum of forces on the mass.

When the mass attains its terminal velocity, this implies the acceleration $\frac{dv}{dt}$ is equal to zero. Let V_T denote the terminal velocity, then

$$0 = m\frac{dv}{dt} = mg - kV_T^3$$

so that

$$V_T = \sqrt[3]{\frac{mg}{k}}.$$

5-33

A large tank initially contains 4 pounds of salt dissolved in 15 gallons of brine. Beginning at time t=0, pure water is poured into the tank at a rate of 2 gallons per minute and the well-stirred mixture is drained out of the tank at a rate of 1 gallon per minute. How many pounds of salt will remain in the tank if this process is allowed to continue for exactly 5 minutes?

****

WE USE $\frac{dx}{dt}$ = "IN" − "OUT" WHERE "IN" AND "OUT" ARE THE RATES AT WHICH SALT ENTERS AND LEAVES THE TANK, RESPECTIVELY.

"IN": { FLOW RATE = 2 GAL/MIN; CONCENTRATION = 0 LB/GAL } $\quad IN = \left(2 \frac{\text{GAL}}{\text{MIN}}\right)\left(0 \frac{\text{LB}}{\text{GAL}}\right) = 0 \frac{\text{LB}}{\text{MIN}}$

"OUT": { FLOW RATE = 1 GAL/MIN; CONCENTRATION = $\frac{x}{15+t}$ LB/GAL } $\quad OUT = \left(1 \frac{\text{GAL}}{\text{MIN}}\right)\left(\frac{x}{15+t} \frac{\text{LB}}{\text{GAL}}\right) = \frac{x}{15+t} \frac{\text{LB}}{\text{MIN}}$

(x = AMOUNT OF SALT IN THE TANK (IN POUNDS) AT TIME t; VOLUME AT TIME t IS $V = 15 + 2t - t = 15 + t$)

$$\Longrightarrow \quad \frac{dx}{dt} = 0 - \frac{x}{15+t} = \frac{-x}{15+t}$$

$$\frac{1}{x}\,dx + \frac{1}{15+t}\,dt = 0$$

$$\int \frac{1}{x}\,dx + \int \frac{1}{15+t}\,dt = \int 0$$

$$\ln|x| + \ln|15+t| = \ln|x(15+t)| = C_1$$

$$x(15+t) = e^{C_1} = C$$

$$x = \frac{C}{15+t}$$

OUR INITIAL CONDITIONS WERE $x = 4$ WHEN $t = 0$, SO

$$4 = \frac{C}{15+0} \Longrightarrow C = 60$$

$$\therefore \quad x = \frac{60}{15+t}$$

WHEN $t = 5$ MIN, $\quad x = \frac{60}{15+5} = \frac{60}{20} = 3$ LB.

5-36

It takes 15 minutes for an object to warm up from 10°C to 20°C in a room whose temperature is 30°C. Assuming Newton's law of cooling, how long would it take to warm up from 20°C to 25 °C?

****

Let $U(t)$ be the temperature of the object at time t. Then by Newton's law of cooling

$$\frac{dU}{dt} = k(30 - U)$$

where $(30-U)$ is the temperature difference and k is the proportional constant. Separation of variables gives

$$\int_{10}^{U} \frac{-dU}{30-U} = \int_0^t -k\,dt$$

incorporating the initial condition $U(0)=10$. Upon integration

$$U(t) = 30 - 20e^{-kt}.$$

How to determine the proportional constant k? We use the other given data: $U(15)=20$. Then

$$20 = 30 - 20e^{-15k}$$

giving

$$k = \frac{\ln 2}{15}.$$

We want to find t such that $U(t)=25$. From the solution

$$25 = 30 - 20e^{-kt}$$

$$kt = \ln 4 = 2\ln 2$$

$$t = \frac{2\ln 2}{\frac{1}{15}\ln 2} = 30.$$

It takes another 15 minutes to warm up from 20°C to 25°C.

5-35

Suppose a ball of mass 2 Kg is thrown upward with an initial velocity 10m/s. It is subjected to the air resistance which is proportional to its instantaneous velocity v. The magnitude of this resistive force is given by 0.5v. Assume gravitational acceleration g = 10m/s^2, find the time taken for the ball to reach its greatest height.

****

The governing differential equation

$$m\frac{dv}{dt} = -mg - R$$

where $v(t)$ is the velocity, m is the mass and R is the resistive force. Putting the numerical constants

$$2\frac{dv}{dt} = -2g - 0.5v = -20 - 0.5v$$

with the initial condition: $v(0)=10$. This is a separable equation

$$\frac{dv}{10+\frac{v}{4}} = -dt.$$

Incorporating the initial condition:

$$\int_{10}^{v}\frac{dv}{10+\frac{v}{4}} = -\int_0^t dt$$

$$v(t) = 50e^{-t/4} - 40.$$

We would like to find t_T such that $v(t_T)=0$.

$$v(t_T) = 50e^{-t_T/4} - 40 = 0$$

Solving for t_T, $t_T = -4\ln 0.8$.

5-38

On a level road, a car is traveling at v_o miles per hour. At time t = 0, the driver shifts into neutral gear and coasts. Assuming the retarding force (rolling friction plus air resistance) is proportional to velocity, determine the distance the car coasts until it comes to rest.

****

Applying Newton's Second Law of Motion $F = ma = m\,dv/dt$, we have $m\frac{dv}{dt} = -kv$, where k is a constant of proportionality.

Separating variables, $\frac{dv}{v} = -\frac{k}{m}dt$.

Integrating, $\ln v = -\frac{k}{m}t + C$, so

$v = e^{-\frac{k}{m}t + C} = e^{C}\cdot e^{-\frac{k}{m}t}$. Since

$v = v_0$ at time $t = 0$, we see that $e^C = v_0$.

Hence the velocity is $V(t) = v_0 e^{-\frac{k}{m}t}$.

The distance travelled is given by

$$S = \int_0^\infty V(t)\,dt = \int_0^\infty v_0 e^{-\frac{k}{m}t}\,dt$$

$$= \left[-\frac{mv_0}{k}e^{-\frac{k}{m}t}\right]_0^\infty = \frac{mv_0}{k}.$$

Thus, the coasting distance is the initial momentum mv_0 divided by the resistance constant.

5-37

A boater and motor boat together weigh 320 pounds. The thrust of the motor produces a constant force of F pounds in the direction of motion, and the water resistance is proportional to the speed of the boat. The boat is initially at rest , then moves in a straight path. Find the velocity of the boat at time t, and show that the limiting velocity gives a net force of zero.

****

$v(t)$ = velocity of boat ; mv' = sum of forces = $F - kv$

where k is a constant of proportionality, and $m = \frac{320}{32} = 10$.

$10v' = F - kv$, or $v' + \frac{k}{10} v = \frac{F}{10}$. This is a linear equation with integrating factor $e^{\int \frac{k}{10}} = e^{\frac{k}{10}t}$. Then

$\left(e^{\frac{k}{10}t} v\right)' = \frac{F}{10} e^{\frac{k}{10}t}$, $e^{\frac{k}{10}t} v = \frac{F}{k} e^{\frac{k}{10}t} + C$, so

$v(t) = \frac{F}{k} + Ce^{-\frac{k}{10}t}$. The boat is initially at rest, so

$v(0) = 0$: $0 = \frac{F}{k} + C$, $C = -\frac{F}{k}$. Thus,

$v(t) = \frac{F}{k} - \frac{F}{k} e^{-\frac{k}{10}t} = \frac{F}{k}\left(1 - e^{-\frac{k}{10}t}\right)$

$\lim_{t \to \infty} v(t) = \frac{F}{k}$ (assuming $k > 0$) and with $v = \frac{F}{k}$

the sum of forces on the boat $F - \frac{k}{v} = 0$.

5-40

A stone is thrown vertically upwards with an initial velocity of $\sqrt{g}$ in a medium offering a resistance of $\sqrt{g}.v$ per unit mass, where v is the velocity of the stone and g is the magnitude of the acceleration due to gravity. if the only other force acting on the stone is gravity, then find the maximum height attained by the stone and the time taken.

****

Let x be the height and v the velocity at time t.
If m is the mass of the stone, then

$$m\frac{dv}{dt} = -mg - m\sqrt{g}\,v$$

i.e. $$\frac{dv}{dt} + \sqrt{g}\,v = -g$$

This linear equation has the solution

$$ve^{\sqrt{g}t} = -\sqrt{g}\,e^{\sqrt{g}t} + C$$

But $v = \sqrt{g}$ at $t = 0$. This gives us $C = 2\sqrt{g}$

Thus $v = \sqrt{g}\left[2e^{-\sqrt{g}t} - 1\right]$.

At the maximum height $v = 0$. If T is the time taken, then

$$2e^{-\sqrt{g}T} - 1 = 0 \quad , \text{i.e. } T = \frac{1}{\sqrt{g}}\ln 2$$

$$\frac{dx}{dt} = v = \sqrt{g}\left[2e^{-\sqrt{g}t} - 1\right].$$

If H is the maximum height, then

$$\int_0^H dx = \int_0^T \sqrt{g}\left[2e^{-\sqrt{g}t} - 1\right]dt$$

$$H = \sqrt{g}\left[\frac{2}{-\sqrt{g}}e^{-\sqrt{g}t} - t\right]_0^{\frac{1}{\sqrt{g}}\ln 2}$$

$$= \sqrt{g}\left[\frac{-2}{\sqrt{g}}e^{-\ln 2} - \frac{1}{\sqrt{g}}\ln 2 + \frac{2}{\sqrt{g}}\right]$$

$$= \left[-\frac{2}{2} - \ln 2 + 2\right] = 1 - \ln 2.$$

5-39

Archimede's Principle states that a body partially or totally submerged in a liquid is buoyed up by a force equal to the weight of the liquid displaced. Let $y = 0$ be the equilibrium position of the lower end of a body when it is floating in a liquid. If the body is depressed and then released, its subsequent motion is governed by the DE:

$$m\frac{d^2y}{dt^2} = -\rho A y$$

Where m is the mass of the body, A is the cross-sectional area of the body and ρ is the density of the liquid.

A body is depressed y_0 feet from its equilibrium position and released.

(a) Show that its equation of motion is: $y = y_0 \cos(\sqrt{\rho A/m})\, t$

(b) If the body is a right circular cylinder with vertical axis and radius r, show that the period of its motion is: $T = \frac{2}{r}\sqrt{\frac{m\pi}{\rho}}$

(c) A cylindrical buoy of radius 1 ft. floats in water ($\rho = 62.5$ lb./cu. ft.). Its period, when depressed and released is 4 sec. What is its weight in pounds?

****

a) The differential equation can be written: $y'' + (\rho A/m)y = 0$. The roots of the auxiliary equation $r^2 + (\rho A/m) = 0$ are $r = \pm i\sqrt{\rho A/m}$. Thus the general solution is given by:

$$y = c_1 \cos\sqrt{\rho A/m}\, t + c_2 \sin\sqrt{\rho A/m}\, t$$

The initial conditions $y(0) = y_0$ and $y'(0) = 0$ imply that $c_1 = y_0$ and $c_2 = 0$.

So, $y = y_0 \cos\sqrt{\rho A/m}\, t$, as desired.

b) From the equation of motion, we know that the period is given by: $T = 2\pi/\sqrt{\rho A/m}$. For the given body, $A = \pi r^2$. Thus,

$$T = \frac{2\pi}{\sqrt{\rho\pi r^2/m}} = \frac{2}{r}\sqrt{\frac{m\pi}{\rho}}, \text{ as desired.}$$

c) Using the result from part (b) we have:

$4 = 2\sqrt{m\pi/62.5} \Rightarrow m = 250/\pi$ slugs

In pounds, weight $= 32(250/\pi) \approx 2546$ lbs.

5-42

A 16 pound object is dropped from a great height with initial velocity $v_0 = 0$. Air resistance exerts a force on the object numerically equal to 1.5v, where v is the instantaneous velocity (in ft/sec) at time t. Find the limiting velocity for this object. (Use $g = 32$ ft/sec^2.)

****

THE WEIGHT HAS MASS $m = \frac{w}{g} = \frac{16}{32} = \frac{1}{2}$ AND HAS TWO FORCES ACTING ON IT :

1) THE FORCE DUE TO GRAVITY (WEIGHT) : $F_1 = 16$
2) THE AIR RESISTANCE : $F_2 = -1.5v$

(NOTE : THE DOWNWARD DIRECTION IS TAKEN AS POSITIVE IN THIS CASE.)

NOW $ma = F$ BECOMES $m\frac{dv}{dt} = F_1 + F_2$

$$\frac{1}{2}\frac{dv}{dt} = 16 - 1.5v$$

$$\frac{dv}{dt} = 32 - 3v$$

WE SEPARATE VARIABLES AND INTEGRATE :

$$\int \frac{dv}{32-3v} = \int dt$$

$$-\frac{1}{3}\ln|32-3v| = t + C_1$$

$$\ln|32-3v| = -3t + C_2$$

$$32-3v = e^{-3t+C_2} = e^{C_2}e^{-3t} = C_3e^{-3t}$$

$$v = \frac{32}{3} + C_4e^{-3t}$$

APPLY INITAL CONDITIONS $v_0 = 0$ TO FIND $C_4 = \frac{-32}{3}$

$$\Longrightarrow \quad v = \frac{32}{3} - \frac{32}{3}e^{-3t}$$

LIMITING VELOCITY :

$$v_\infty = \lim_{t\to\infty}\left(\frac{32}{3} - \frac{32}{3}e^{-3t}\right) = \frac{32}{3} - 0 = \frac{32}{3} \text{ FT/SEC}$$

5-41

An object is projected vertically downward, from a great height, with velocity v_0. The resistance is proportional to the magnitude of the velocity.

(a) Find the velocity as a function of time t

(b) Find the limiting velocity

and (c) Find the time before the object reaches 95% of its limiting value.

****

(a) $ma = F$ (Newton's 2nd Law)

$m \cdot \frac{dv}{dt} = mg - Kv$

$\Rightarrow \frac{dv}{dt} + \frac{K}{m} v = g$; 1st order linear and $\mu = e^{\int \frac{K}{m} dt} = e^{\frac{K}{m} t}$

$e^{\frac{K}{m}t}\left[\frac{dv}{dt} + \frac{K}{m} v\right] = g\, e^{\frac{K}{m}t}$

$\Rightarrow \frac{d}{dt}\left[e^{\frac{K}{m}t} \cdot v\right] = g e^{\frac{K}{m}t} \Rightarrow e^{\frac{K}{m}t} \cdot v = \frac{mg}{K} e^{\frac{K}{m}t} + C$

$\Rightarrow v = \frac{mg}{K} + C e^{-\frac{K}{m}t}$ $\quad v(0) = v_0 \Rightarrow v_0 = v(0) = \frac{mg}{K} + C \cdot 1$

$\Rightarrow C = v_0 - \frac{mg}{K}$ $\quad \therefore v(t) = \frac{mg}{K} + \left(v_0 - \frac{mg}{K}\right) e^{-\frac{K}{m}t}$ (*)

(b) as $t \to +\infty$, $v(t) \to \frac{mg}{K}$ = limiting velocity

(c) Dividing (*) by $\frac{mg}{K}$ $\Rightarrow (.95 =) \frac{v}{\left(\frac{mg}{K}\right)} = 1 + \left[\frac{v_0}{\left(\frac{mg}{K}\right)} - 1\right] e^{-\frac{K}{m}t}$

$\Rightarrow -.05 = \left[\frac{v_0 K}{mg} - 1\right] e^{-\frac{K}{m}t} \Rightarrow .05 = \left[1 - \frac{v_0 K}{mg}\right] e^{-\frac{K}{m}t}$

$\Rightarrow .05 = \left[\frac{mg - v_0 K}{mg}\right] e^{-\frac{K}{m}t}$

$\Rightarrow .05 \left[\frac{mg}{mg - v_0 K}\right] = e^{-\frac{K}{m}t}$

$\Rightarrow \ln\left\{.05\left[\frac{mg}{mg - v_0 K}\right]\right\} = -\frac{K}{m} t \Rightarrow t = -\frac{m}{K} \ln\left\{.05\left[\frac{mg}{mg - v_0 K}\right]\right\}$

[Note; if dropped from rest, $v_0 = 0$ and $t = -\frac{m}{K} \ln(.05)$]

5-44

A spring mass system is governed by the equation y" - y = 15 sin t
What are the steady state and transient parts of its solution?

****

the complementary solution has characteristic equation

$$m^2 - 1 = 0 \quad \text{or} \quad m = \pm 1$$

so, complementary solution is $y = C_1 e^t + C_2 e^{-t}$

To obtain a particular solution, assume

$$y = A \sin t + B \cos t$$

therefore $y'' = -A \sin t - B \cos t$

substituting in the nonhomogeneous equation we obtain

$$y'' - y = 15 \sin t$$

$$-A \sin t - B \cos t - A \sin t - B \cos t = 15 \sin t$$

$$-2A \sin t - 2B \cos t = 15 \sin t$$

Equating coefficients of similar terms we obtain (note sin t and cos t are independent functions)

$$B = 0, \quad A = -\frac{15}{2}$$

Therefore general solution of differential equation is

$$y = C_1 e^t + C_2 e^{-t} - \frac{15}{2} \sin t$$

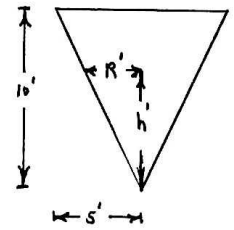
as $t \to \infty$, the term $C_2 e^{-t} \to 0$. so steady state solution is $y = C_1 e^t - \frac{15}{2} \sin t$ and the transient solution is $y = C_2 e^{-t}$

5-43

An open tank is in the form of a right circular cone with vertex down. The radius of the tank is five feet and the altitude is ten feet. If water evaporates from the tank at a rate which is proportional to the exposed surface area and the depth decreases from ten feet to five feet in twenty days, when will the depth be two feet?

****

IN ANY "STATEMENT" PROBLEM WHERE A DIAGRAM CAN BE DRAWN, DRAW ONE AND LABEL ALL OF THE PARTS CLEARLY. FROM THIS ONE, AT ANY FLUID LEVEL OF h', THE RADIUS IS R' AND THE VOLUME OF FLUID IS



$V = \frac{1}{3}\pi R^2 h$. THE SURFACE AREA OF THE FLUID IN THE CONE IS A CIRCLE, SO THAT AREA, A, IS πR^2.

FROM SIMILAR TRIANGLES, $\frac{5}{10} = \frac{R}{h}$ OR $R = \frac{h}{2}$.

WATER EVAPORATING CORRESPONDS TO THE RATE OF CHANGE OF VOLUME WITH RESPECT TO TIME, t IN DAYS. THIS IS $\frac{dV}{dt}$. IT IS ALSO GIVEN THAT $\frac{dV}{dt}$ IS PROPORTIONAL TO A, SO WE WRITE $\frac{dV}{dt} = KA$.

THEN $V = \frac{1}{3}\pi R^2 h = \frac{1}{3}\pi\left(\frac{h}{2}\right)^2 h = \frac{\pi}{12}h^3$.

$\therefore \frac{dV}{dt} = \frac{d}{dt}\left(\frac{\pi}{12}h^3\right) = \frac{\pi}{12}\cdot 3h^2\frac{dh}{dt} = \frac{\pi}{4}h^2\frac{dh}{dt}$.

IF $\frac{dV}{dt} = KA$, THEN $\frac{\pi}{4}h^2\frac{dh}{dt} = K\pi R^2 = K\pi\left(\frac{h}{2}\right)^2 = \frac{K\pi}{4}h^2$.

$\therefore$ AFTER DIVISION, $\frac{dh}{dt} = K$, FROM WHICH $dh = K\,dt$ OR $h = Kt + C$.

WHEN $t = 0$, $h = 10$. $\therefore 10 = K(0) + C$. $C = 10$, AND $h = Kt + 10$.

WHEN $t = 20$, $h = 5$. $\therefore 5 = K(20) + 10$. $K = -\frac{1}{4}$ AND $h = -\frac{1}{4}t + 10$.

THE RELATION BETWEEN h AND t IS NOW KNOWN. FOR $h = 2'$, $2 = -\frac{1}{4}t + 10$ AND $t = 32$ DAYS.

5-46

Unafraid-to-jump Sally, weighing 160 pounds with full gear, bails out of an airplane at a great height. If air resistance during her descent is proportional to her velocity of motion, with resistance = K•V ≡ 1•V ; V > 0, determine the height of her fall and also the velocity of her impact with the ground, if she touches down after 60 seconds.

Herein apply the second law of Newtonian mechanics, and assume that $g = 32$ ft./$_{sec.}{}^2$ is the 'gravity' constant; also assume that V = 0 at t = 0 ; and that the 'chute opens immediately after she bails out of the plane.

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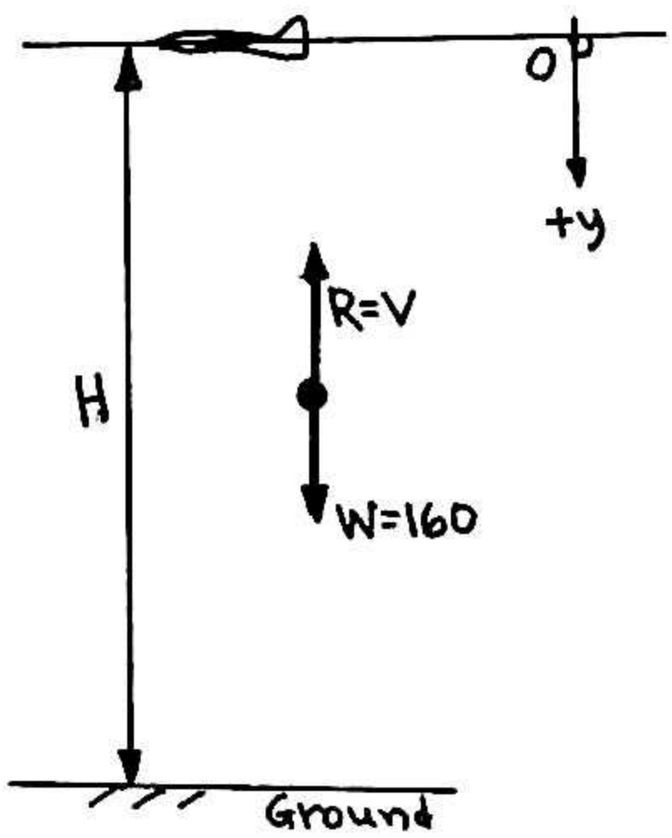
Referring to the sketch below, and summing forces in the direction of motion, Newton's Law gives:



$$\underline{F = m \cdot a} \text{ , or } \underline{W - R = m \cdot a}.$$

Here, $\underline{160 - V = \frac{W}{g} \cdot a = \frac{160}{32} \cdot \frac{dV}{dt}}$.

This first order differential equation is separable, giving

$$\int \frac{5}{160 - V} \cdot dV = \int dt \text{ , or}$$

$$\underline{-5 \cdot \ln|160 - V| = t + c_1}.$$

V=0 at t=0 gives $c_1 = -5 \cdot \ln 160$. Thus the solution (relating velocity and time) can be expressed as:

$$\underline{t = -5 \cdot \ln(160 - V) + 5 \cdot \ln(160) \equiv -5 \cdot \ln\left(\frac{160 - V}{160}\right)}.$$

Via logarithms, we write $e^{-\frac{1}{5}t} = 1 - \frac{V}{160}$.

Thus, $\underline{V = V(t) \equiv \frac{dy}{dt} = 160 \cdot (1 - e^{-\frac{1}{5}t})}$

5-45

If an undamped spring-mass system with a mass that weighs 32 lb and a spring constant 1 lb/ft is suddenly set in motion at t = 0 by an external force of cost lb, determine the subsequent motion and draw a graph of the displacement versus t.

****

$m = \frac{32}{32} = 1$, $k = 1$, $F_0 = \cos t$, so

$$\ddot{u} + u = \cos t, \quad u(0) = 0, \quad \dot{u}(0) = 0$$

First find a particular solution of $\ddot{u} + u = \cos t$, set

$$u_p(t) = t\,(A\cos t + B\sin t)$$

$$\dot{u}_p(t) = A\cos t + B\sin t - t\,(A\sin t - B\cos t)$$

$$\ddot{u}_p(t) = -2A\sin t + 2B\cos t - t\,(A\cos t + B\sin t)$$

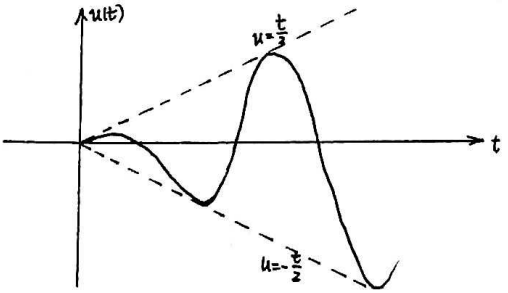
so $\quad \cos t = -2A\sin t + 2B\cos t$

$$A = 0, \quad B = \frac{1}{2}, \quad u_p(t) = \frac{t}{2}\sin t$$

general solution $u(t) = A\cos t + B\sin t + \frac{t}{2}\sin t$

$u(0) = A = 0$, $\dot{u}(0) = B = 0$, so

$$u(t) = \frac{t}{2}\sin t$$



5-47

A bullet is fired with an initial velocity 200 m/s against a wooden block of thickness h = 0.1m. It comes out from the block with remaining velocity 80 m/s. Assume the resistance to the bullet by the block is proportional to the square of the velocity.

Find the time required for the bullet to travel through the block.

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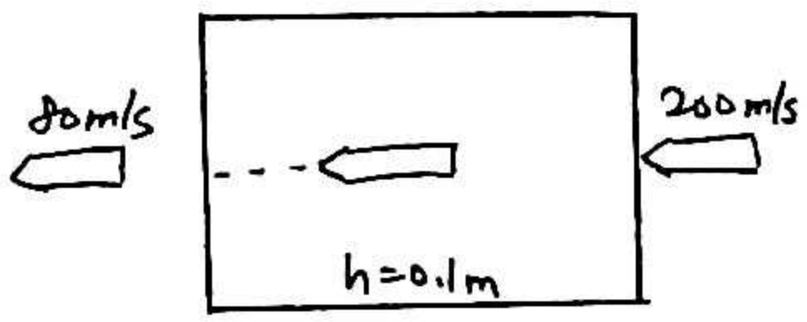
Let m be the mass of the bullet, the resistance be kV^2 where $V(t)$ is the velocity of the bullet.



Then

$$m\frac{dV}{dt} = -kV^2 .$$

Setting $a^2 = k/m$, the equation becomes

$$\frac{dV}{dt} = -a^2V^2 .$$

Using separation of variables

$$\frac{dV}{-V^2} = a^2dt$$

and upon integration

$$\frac{1}{V} = a^2t + C_1 .$$

To determine C_1, we use the initial condition: $V(0) = 200$.

Therefore

$$\frac{1}{V} = a^2t + \frac{1}{200} .$$

Let T be the time required for the bullet to travel through the block; at $t = T$, $V(T) = 80$. Hence

$$\frac{1}{80} = a^2T + \frac{1}{200}$$

Again, separation of variables gives

$$y = \int V \cdot dt = 160 \cdot \left[t + 5 \cdot e^{-\frac{1}{5}t} \right] + c_2 .$$

$y = 0$ at $t = 0$ gives $c_2 = -800$.

Thus $$y = y(t) = 160 \cdot \left[-5 + t + 5 \cdot e^{-\frac{1}{5}t} \right] .$$

Now $V_{IMPACT} = V(60) = 160 \cdot (1 - e^{-12}) \approx 160.$ ft/sec = 'terminal' velocity,

and H = total height of fall $= y(60) = 160 \cdot [-5 + 60 + 5 \cdot e^{-12}]$

$\approx 8{,}800.$ ft.

Via second order techniques, Newton's second law reads

$$160 - \dot{y} = 5 \cdot \ddot{y} \; ; \qquad (\dot{}) \equiv \frac{d}{dt}( )$$

The solution of this non-homogeneous, linear, second order ordinary differential equation, with constant coefficients,

$5\ddot{y} + \dot{y} = 160$ is $y = y_{CF} + y_{PI}$;

y_{CF} = complimentary function $= \hat{c}_1 + \hat{c}_2 \cdot e^{-\frac{1}{5}t}$, and

y_{PI} = particular integral $= 160 \cdot t$.

Thus $$y = 160 \cdot t + \hat{c}_1 + \hat{c}_2 \cdot e^{-\frac{1}{5}t} .$$

From initial conditions, $y = 0$, $V = 0$, at $t = 0$, $\hat{c}_1$ and $\hat{c}_2$ are determined, giving

$$y = 160 \cdot \left[-5 + t + 5 \cdot e^{-\frac{1}{5}t} \right] .$$

5-48

An adventuring mathematics professor leaps from a plane, immediately opening his parachute. He attaches a second parachute that opens automatically to his crate of books. The crate has a mass of 180 kg, twice that of the professor. If the crate has a terminal velocity of 5 m/s, what is the professor's terminal velocity? How long until the professor reaches half of his terminal velocity? Assume that the two parachutes are identical, so that their constants of proportionality are the same.

****

total force = force due to gravity − resistive force of air

$ma = mg - kv$ $\quad$ k being a constant

at terminal velocity, acceleration is zero, so $0 = mg - kv$

thus $v_T = \frac{mg}{k}$

crate: $m = 180\ kg$, $v_T = 5\ m/s$, $g = 9.8\ m/s^2$

$$k = \frac{mg}{v_T} = \frac{(180\ kg)(9.8\ m/s^2)}{5\ m/s} = 352.8\ kg/s$$

professor: $m = 90\ kg$, $g = 9.8\ m/s^2$, $k = 352.8\ kg/s$

$$v_T = \frac{mg}{k} = \frac{(90\ kg)(9.8\ m/s^2)}{352.8\ kg/s} = 2.5\ m/s$$

need the velocity function to find when velocity is $\frac{1}{2} v_T$:

$$m\frac{dv}{dt} = mg - kv$$

$$\frac{dv}{dt} + \frac{k}{m} v = g$$ $\quad$ first order linear

$$\mu(t) = e^{\int \frac{k}{m} dt} = e^{\frac{k}{m} t}$$

which gives $$a^2 = \frac{3}{400T}.$$

The velocity can be written as

$$V(t) = \frac{400T}{3t+2T}.$$

Let $h(t)$ be the penetration depth inside the block. Then

$$V(t) = \frac{dh(t)}{dt} = \frac{400T}{3t+2T}$$

and

$$\frac{dh}{400T} = \frac{dt}{3t+2T}.$$

Using the initial condition: $h(0) = 0$,

$$\int_0^h \frac{dh}{400T} = \int_0^t \frac{dt}{3t+2T}$$

giving

$$\frac{h(t)}{400T} = \frac{1}{3}\ln\left(\frac{3t+2T}{2T}\right).$$

Our final goal is to find T. This can be done by using the fact that $h(T) = 0.1$. Therefore

$$\frac{0.1}{400T} = \frac{1}{3}\ln\left(\frac{5T}{2T}\right)$$

which gives

$$T = \frac{3}{4000\ln(5/2)}.$$

5-49

An 18 pound weight is on a spring which has a spring constant of $\frac{27}{8}$ and a damping constant of $\frac{9}{4}$. If the weight is given a downward velocity of $2\sqrt{2}$ feet per second from the equilibrium position find:
a) the position at any time.
b) the quasiperiod and time varying amplitude.
c) the time that the weight passes through equilibrium going upwards.

****

a) $\frac{18}{32}x'' + \frac{9}{4}x' + \frac{27}{8}x = 0$ $\quad x(0) = 0$, $v(0) = 2\sqrt{2}$

$x'' + 4x' + 6x = 0$

$m^2 + 4m + 6 = 0 \Rightarrow m = \frac{-4 \pm \sqrt{16 - 4\cdot 6}}{2}$

$m = \frac{-4 \pm 2\sqrt{2}\,i}{2}$

$m = -2 \pm \sqrt{2}\,i$

$x = e^{-2t}(c_1 \sin\sqrt{2}t + c_2 \cos\sqrt{2}t)$

$x(0) = 0 = e^0(c_1 \cdot 0 + c_2 \cdot 1) \Rightarrow c_2 = 0$

$x = c_1 e^{-2t}\sin\sqrt{2}t$

$v = \frac{dx}{dt} = -2c_1 e^{-2t}\sin\sqrt{2}t + \sqrt{2}c_1 e^{-2t}\cos\sqrt{2}t$

$v(0) = 2\sqrt{2} = -2c_1 \cdot e^0 \cdot 0 + \sqrt{2}c_1 e^0 \cdot 1 \Rightarrow c_1 = 2$

$\underline{\underline{x = 2e^{-2t}\sin\sqrt{2}t}}$

b) the quasiperiod is $\underline{\underline{\frac{2\pi}{\sqrt{2}}}}$

the time varying amplitude is $\underline{\underline{2e^{-2t}}}$

c) the weight passes through equilibrium when $x(t) = 0$. ie. when

$2e^{-2t}\sin\sqrt{2}t = 0$

Since $e^{-2t} \neq 0 \Rightarrow \sin\sqrt{2}t = 0$.

$$e^{\frac{k}{m}t}\frac{dV}{dt} + \frac{k}{m}e^{\frac{k}{m}t}V = g e^{\frac{k}{m}t}$$

$$\frac{d}{dt}\left(e^{\frac{k}{m}t}V\right) = g e^{\frac{k}{m}t}$$

$$e^{\frac{k}{m}t}V = \frac{gm}{k}e^{\frac{k}{m}t} + C$$

$$V = \frac{gm}{k} + C e^{-\frac{k}{m}t}$$

using $v(0) = 0$: $\quad 0 = \frac{gm}{k} + C \qquad C = -\frac{gm}{k}$

so $\quad v(t) = \frac{gm}{k}\left(1 - e^{-\frac{k}{m}t}\right)$

notice: $v_T = \lim_{t\to\infty} v(t) = \lim_{t\to\infty} \frac{gm}{k}\left(1 - e^{-\frac{k}{m}t}\right) = \frac{gm}{k}$, as before

professor: $\quad v(t) = 2.5\left(1 - e^{-3.92t}\right)$

$$\tfrac{1}{2}(2.5) = 1 - e^{-3.92t} \quad \Rightarrow \quad t = 0.177 \text{ s}$$

So after 0.177 seconds, his velocity will be half his terminal velocity.

HARMONIC MOTION

5-50

A 0.05 Kg mass can stretch a spring 0.025 m under its own weight. The spring satisfies Hooke's law and there is no friction or external force. Determine the general solution and the frequency with which the mass will oscillate. Suppose we further know that the spring mass system is oscillating with an amplitude of 0.05m. Find the maximum velocity. Assume g = 10 m/s^2, g is the gravitational acceleration.

****

By Hooke's law : force $\propto$ extension. Let K be the spring constant, then

weight = $mg = k\Delta x$ where Δx is the extension.

$$0.05 \times 10 = K(0.025)$$

$$K = 20 .$$

The governing differential equation of oscillation

$$m\frac{d^2x}{dt^2} + Kx = 0$$

where $X(t)$ is the displacement of the mass from equilibrium position. The solution for the motion

$$X(t) = A\cos\omega t + B\sin\omega t$$

where ω is the angular frequency of oscillation. We have

$$\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{20}{0.05}} = 20$$

$\sin \sqrt{2}t = 0$ when $\sqrt{2}t = n\pi$ $n = 0, \pm 1, \pm 2 \ldots$

or $t = \frac{n\pi}{\sqrt{2}}$

the weight will be going downward

$t = 0, \frac{2\pi}{\sqrt{2}}, \frac{4\pi}{\sqrt{2}}, \frac{6\pi}{\sqrt{2}}, \ldots$

[Note that the weight will be going upward when $t = \frac{\pi}{\sqrt{2}}, \frac{3\pi}{\sqrt{2}}, \frac{5\pi}{\sqrt{2}}, \ldots$]

5-51

A 16 pound weight stretches a spring 6 inches. The spring is then compressed 1 foot and released with a downward velocity of 3 ft/sec. An external force of 2sinwt pounds is applied. Neglect damping. (a) For what value of w would resonance occur? (b) Find the motion of the spring in the non-resonance case.

****

(a) The spring constant is $k = \frac{16}{1/2} = 32$ and the mass is $m = \frac{16}{32} = \frac{1}{2}$. Resonance occurs if $\omega = \sqrt{\frac{k}{m}} = \sqrt{64} = 8$

(b) The equation is $\frac{1}{2}x'' + 32x = 2\sin\omega t$, $\omega \neq 8$, or

$x'' + 64x = 4\sin\omega t$; initial conditions are $x(0) = -1$, $x'(0) = 3$.

The characteristic equation is $r^2 + 64 = 0$, $r = \pm 8j$, so the general solution of the homogeneous equation is

$y_c = c_1 \sin 8t + c_2 \cos 8t$.

If $\omega \neq 8$, a particular solution has the form $x_p = A\sin\omega t + B\cos\omega t$

Then $x_p'' + 64x_p = (64-\omega^2)A\sin\omega t + (64-\omega^2)B\sin\omega t = 4\sin\omega t$

$$A = \frac{4}{64-\omega^2} \qquad B = 0$$

The general solution is $x(t) = c_1 \sin 8t + c_2 \cos 8t + \frac{4}{64-\omega^2}\sin\omega t$

$$x(0) = c_2 = -1$$

$$x'(0) = 8c_1 + \frac{4\omega}{64-\omega^2} = 3 \quad ; \quad c_1 = 3 - \frac{4\omega}{64-\omega^2}$$

$$x(t) = \left(3 - \frac{4\omega}{64-\omega^2}\right)\sin 8t - \cos 8t + \frac{4}{64-\omega^2}\sin\omega t$$

so that

$$X(t) = A\cos 20t + B\sin 20t.$$

Assume the following initial conditions: at $t=0$, the mass attains its maximum amplitude. The initial conditions are then $X(0) = 0.05$ and $X'(0) = 0$. Now

$$X(0) = A = 0.05$$

and

$$X'(t) = \omega B \cos \omega t - \omega A \sin \omega t.$$

To find the other constant B, we enforce $X'(0) = 0$. This gives

$$X'(0) = B = 0.$$

Hence the solution for the oscillatory motion

$$X(t) = 0.05 \cos 20t,$$

and the velocity $X'(t)$ is given by

$$X'(t) = -\sin 20t.$$

The maximum velocity

$$\max |X'(t)| = \max |-\sin 20t| = 1.$$

Remark: A simple formula to remember

maximum velocity = amplitude X angular frequency

used

5-53

A 1-slug mass is attached to a spring whose Hooke constant is 16 lb/ft. The system is immersed in a resistive medium causing a retarding force proportional to velocity, with proportionality constant of 10. After coming to rest in equilibrium position, the mass is subject to a force of 750sin(4t) pounds. Determine the subsequent motion of the mass.

****

Let y denote deviation from equilibrium. The restoring force is $-16y$ and the retarding force is $-10y'$. Applying Newton's Second Law of Motion $ma = \sum F$, we have $my'' = -10y' - 16y + 750\sin 4t$. With $m = 1$, we get the equation

$$y'' + 10y' + 16y = 750\sin 4t.$$

The homogeneous equation has associated quadratic equation $r^2 + 10r + 16 = 0$, with roots $r_1 = -2$ and $r_2 = -8$. Hence the solution of the homogeneous equation is

$$y_h = c_1 e^{-2t} + c_2 e^{-8t}.$$

Using the Method of Undetermined Coefficients to find a particular solution, we try

$y_p = A\cos 4t + B\sin 4t$, with derivatives

$y_p' = -4A\sin 4t + 4B\cos 4t$ and

$y_p'' = -16A\cos 4t - 16B\sin 4t.$

5-52

An 8 pound weight stretches a spring 16 inches beyond its original length when attached. At time t=0 the weight is pulled 1 foot below the equilibrium position and released with initial velocity $v_0=0$. If the medium offers a resistance numerically equal to twice the instantaneous velocity of the object, find the displacement as a function of t and determine the quasi period of this damped oscillation.

****

WEIGHT $w = 8$ lb. $\Rightarrow$ MASS $m = \frac{w}{g} = \frac{8}{32} = \frac{1}{4}$ SLUG

DAMPING CONSTANT $a = 2$

HOOKE'S LAW WITH $s = 16'' = \frac{4}{3}'$ $\Rightarrow$ SPRING CONSTANT $k = \frac{w}{s} = \frac{8}{4/3} = 6$

INITIAL CONDITIONS: $t=0$, $x_0 = 1'$, $v_0 = 0$

$$m\frac{d^2x}{dt^2} + a\frac{dx}{dt} + kx = 0$$

$$\frac{1}{4}\frac{d^2x}{dt^2} + 2\frac{dx}{dt} + 6x = 0$$

$$\frac{d^2x}{dt^2} + 8\frac{dx}{dt} + 24x = 0$$

THE CHARACTERISTIC EQUATION HAS ROOTS $\frac{-8 \pm \sqrt{64-96}}{2} = \frac{-8 \pm \sqrt{-32}}{2}$

$$= -4 \pm 2\sqrt{2}\,i$$

$$\Rightarrow x(t) = \left(c_1 \cos 2\sqrt{2}\,t + c_2 \sin 2\sqrt{2}\,t\right) e^{-4t}$$

$$v(t) = \left[\left(-4c_1 + 2\sqrt{2}\,c_2\right)\cos 2\sqrt{2}\,t + \left(-4c_2 - 2\sqrt{2}\,c_1 \sin 2\sqrt{2}\,t\right)\right] e^{-4t}$$

INITIAL CONDITIONS $\Rightarrow$ $\begin{cases} 1 = c_1 \\ 0 = -4c_1 + 2\sqrt{2}\,c_2 \end{cases}$ $\therefore c_2 = \sqrt{2}$

$$x(t) = \left(\cos 2\sqrt{2}\,t + \sqrt{2}\sin 2\sqrt{2}\,t\right) e^{-4t}$$

$$= \sqrt{3}\left(\frac{1}{\sqrt{3}}\cos 2\sqrt{2}\,t + \frac{\sqrt{2}}{\sqrt{3}}\sin 2\sqrt{2}\,t\right) e^{-4t}$$

$$= \sqrt{3}\cos\left(2\sqrt{2}\,t - \phi\right) e^{-4t}$$

WHERE $\begin{cases} \cos\phi = 1/\sqrt{3} \\ \sin\phi = \sqrt{2}/\sqrt{3} \end{cases}$ $\phi = .9553$ RAD.

QUASI PERIOD $= \frac{2\pi}{2\sqrt{2}} = \frac{\pi}{\sqrt{2}}$ SECONDS

5-54

A weight of 8 lb is attached to a steel spring of length 15 in. The weight extends the spring a distance of 1/8 ft. If the weight is also attached to an oil dashpot with a damping constant c, determine the values of c for which the motion will be overdamped, critically damped and underdamped. Be sure to give units for c.

****

By Hooke's law : $w = k\,\Delta l$, k is the spring constant.

Now $w = 8$ and $\Delta l = 1/8$ so that $k = 64$.

Mass of the object $= 8/32 = 1/4$. Dynamic equation of the spring mass system

$$\frac{1}{4}\ddot{x} + c\dot{x} + 64x = 0$$

where $c\dot{x}$ is the damping term (c must be positive).

The auxillary equation

$$\frac{1}{4}\lambda^2 + c\lambda + 64 = 0$$

or

$$\lambda^2 + 4c\lambda + 256 = 0.$$

The motion is overdamped, critically damped or underdamped according to whether the discriminant

$$(4c)^2 - 4(256)$$

is positive, zero or negative respectively. Thus we have

(i) overdamped $(4c)^2 - 4(256) > 0$ giving $c > 8$,

(ii) critically damped : $c = 8$; and

(iii) underdamped : $c < 8$.

Unit of c is 16 sec/ft.

Substituting these into the D.E. and collecting cosine terms, we see that the coefficients satisfy $-16A + 40B + 16A = 0$, so $B = 0$. Collecting sine terms with $B = 0$, we get just $-40A = 750$, so $A = -18.75$.

Hence the general solution is

$$y_g = y_h + y_p = c_1 e^{-2t} + c_2 e^{-8t} - 18.75\cos 4t$$

with derivative $y_g' = -2c_1 e^{-2t} - 8c_2 e^{-8t} + 75\sin 4t$.

The "rest" initial conditions at $t = 0$ are of course $y(0) = 0$, $y'(0) = 0$. Substituting into y_g and y_g' gives $c_1 + c_2 = 18.75$

and $-2c_1 - 8c_2 = 0$.

These easily solve to get $c_1 = 25$ and $c_2 = -6.25$. Thus, the subsequent deviation from equilibrium is given by

$$y(t) = 25e^{-2t} - 6.25e^{-8t} - 18.75\cos 4t.$$

5-56

Consider a spring fixed at its upper end and supporting a weight of 10 pounds at the lower end. Suppose the 10 lb. weight stretches the spring by 6 inches. (Assume g = 32).

(a) Find the equation of motion of the weight if it is drawn to a position 4 inches below its equilibrium position and released.

(b) As in (a), except assume a damping constant, c = 5 $\frac{\text{lb - sec}}{\text{ft}}$

(c) As in (a), except assume a damping constant, c = 4 $\frac{\text{lb - sec}}{\text{ft}}$

and sketch a rough graph of the solution.

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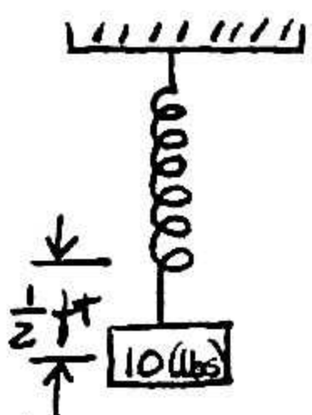


(Restoring force $F = K \cdot X$)

(a) $my'' + cy' + Ky = 0$ (c = damping const.)

(y = displacement) K = Spring Constant

$\therefore y'' + \frac{K}{m} y = 0$

Note: At Equilibrium $F = W = K(\frac{1}{2})$

$y'' + \frac{2W \cdot g}{W} y = 0$ $\Rightarrow K = 2W$, $(W = 10, g = 32)$

or $y'' + (2g)y = 0 \Rightarrow y'' + 64y = 0$

$r^2 + 64 = 0$

$\Rightarrow r = \pm 8i$

$\therefore$ y (the displacement from its equil. length) has the form $y = c_1 \cos 8t + c_2 \sin 8t$

but $\frac{1}{3} = y(0)$, $\Rightarrow \frac{1}{3} = y(0) = c_1 \cdot 1 + c_2 \cdot 0 \Rightarrow \underline{c_1 = 1/3}$

Released from rest $\Rightarrow y'(0) = 0 = 0 + 8c_2 \cdot 1 \Rightarrow \underline{c_2 = 0}$

$\therefore \underline{\underline{y(t) = \frac{1}{3} \cos 8t}}$

(b) $y'' + \frac{c}{m} y' + \frac{k}{m} y = 0 \quad c = 5$

$\Rightarrow y'' + \frac{5g}{W} y' + \frac{2W}{W} g\, y = 0$

$\Rightarrow y'' + 16y' + 64y = 0$

$\Rightarrow r^2 + 16r + 64 = 0$

$(r + 8)^2 = 0 \Rightarrow r = -8$, mult. 2

$\Rightarrow y = c_1 e^{-8t} + c_2 t e^{-8t} = e^{-8t}[c_1 + c_2 t]$

$\frac{1}{3} = y(0) = c_1 \quad 0 = y'(0) = -8c_1 + c_2 \cdot 1 \Rightarrow c_2 = \frac{8}{3}$

$\therefore \underline{\underline{y(t) = e^{-8t}\left[\frac{1}{3} + \frac{8}{3} t\right]}}$

used

5-55

A 32-lb weight is hung from a spring with a spring constant of $k = 16$. The damping constant is $\beta = 8$. The weight is started in motion by displacing it 1 ft. above the equilibrium position with an initial velocity of 2 ft/sec upward.

(a) Write the initial value problem.

(b) Solve it to find the equation of motion.

(c) When does the weight reach its maximum displacement from equilibrium?

(d) When does the weight pass through the equilibrium position?

****

a) Replacing the constants in the general equation for damped vibrations with those from the problem we have:

$$\frac{d^2x}{dt^2} + 8\frac{dx}{dt} + 16x = 0$$

with initial conditions: $x(0) = -1$, $x'(0) = -2$.

b) From the auxiliary equation $r^2 + 8r + 16 = 0$ we have the general solution:

$$x(t) = c_1e^{-4t} + c_2te^{-4t}$$

Now, $x(0) = -1 \Rightarrow c_1 = -1$ and $x'(0) = -2 \Rightarrow c_2 = -6$.
Thus the equation of motion is

$$\underline{\underline{x(t) = -e^{-4t} - 6te^{-4t}.}}$$

c) The maximum displacement occurs when $x'(t) = 0$.

Now, $x'(t) = -2e^{-4t} + 24te^{-4t}$ and $x'(t) = 0 \Rightarrow \underline{\underline{t = 1/12 \text{ sec.}}}$

d) The weight passes through the equilibrium position when $x(t) = 0$. Now, $x(t) = 0 \Rightarrow t = -1/6$ sec.
Thus the weight <u>DOES NOT</u> pass through the equilibrium position.

5-57

$mx'' + kx = g\cos t$. m, k, g, are constants. A particular solution is

(A) $(g/[k-m])\cos t$

(B) $(g/[k-m])\sin t + (k/m)\cos t$

(C) $(gk/[m-k])\sin t$

(D) $(gk/m)\sin(k/m)^{\frac{1}{2}}t + (g/mk)\cos(k/m)^{\frac{1}{2}}t$

(E) $(g/km) + (k/m)\sin(k/m)^{\frac{1}{2}}t$

****

Dividing by m,

$$x'' + \frac{k}{m}x = \frac{g}{m}\cos t$$

Denoting $D \equiv \frac{d}{dt}$

$$(*)\quad \left(D^2 + \frac{k}{m}\right)x = \frac{g}{m}\cdot\frac{1}{2}\left(e^{it} + e^{-it}\right)$$

We seek an equation that equals zero. To that end note that e^{it} is annihilated by the operator $(D-i)$ i.e. $(D-i)e^{it} = 0$, while e^{-it} is annihilated by $(D+i)$. To annihilate the sum $e^{it} + e^{-it}$, we use the composition operator. In fact, applying $(D-i)(D+i)$ to both sides of the equation (*), the right hand side of the equation vanishes since

$$\underbrace{(D-i)(D+i)}_{(D+i)(D-i)}e^{it} + (D-i)\underbrace{(D+i)e^{-it}}_{0} = 0.$$

and the equation becomes

$$(D-i)(D+i)\left(D^2 + \frac{k}{m}\right)x = 0.$$

To solve it, put $x(t) = e^{\lambda t}$ (where λ is some number). Then,

$$(\lambda-i)(\lambda+i)\left(\lambda^2 + \frac{k}{m}\right)e^{\lambda t} = 0$$

(c) $y'' + \frac{c}{m}y' + \frac{k}{m}y = 0 \qquad c = 4$

$y'' + \frac{4g}{W}y' + \frac{2W}{W}g\,y = 0$

$\Rightarrow y'' + \frac{64}{5}y' + 64y = 0$

$\sqrt{\frac{(-36)(64)}{25}} = \frac{6\cdot 8}{5}i$

$\sqrt{\frac{64^2 - 4(64)(25)}{25}}$

$r^2 + \frac{64}{5}r + 64 = 0 \Rightarrow r = \frac{-\frac{64}{5} \pm \sqrt{\left(\frac{64}{5}\right)^2 - 4(64)}}{2} = -\frac{32}{5} \pm \frac{24}{5}i$

$\therefore e^{-\frac{32}{5}t}\cos\frac{24}{5}t,\ e^{-\frac{32}{5}t}\sin\frac{24}{5}t$ are 2 linearly independent solutions.

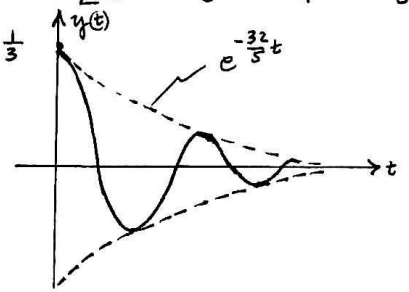
$\therefore y = c_1 e^{-\frac{32}{5}t}\cos\frac{24}{5}t + c_2 e^{-\frac{32}{5}t}\sin\frac{24}{5}t$

$\frac{1}{3} = y(0) = c_1 \quad ; \quad 0 = y'(0) = c_1\left(-\frac{32}{5}\right) + c_2\left(\frac{24}{5}\right)$

$\Rightarrow c_2\left(\frac{24}{5}\right) = \left(\frac{32}{5}\right)\left(\frac{1}{3}\right) \Rightarrow c_2 = \left(\frac{8}{24}\right)\left(\frac{32}{5}\right)\left(\frac{1}{3}\right) = \frac{4}{9}$

$\therefore y(t) = \frac{1}{3}e^{-\frac{32}{5}t}\cos\frac{24}{5}t + \frac{4}{9}e^{-\frac{32}{5}t}\sin\frac{24}{5}t$

or $y(t) = e^{-\frac{32}{5}t}\left[\frac{1}{3}\cos\frac{24}{5}t + \frac{4}{9}\sin\frac{24}{5}t\right]$



ELECTRIC CIRCUITS

5-58

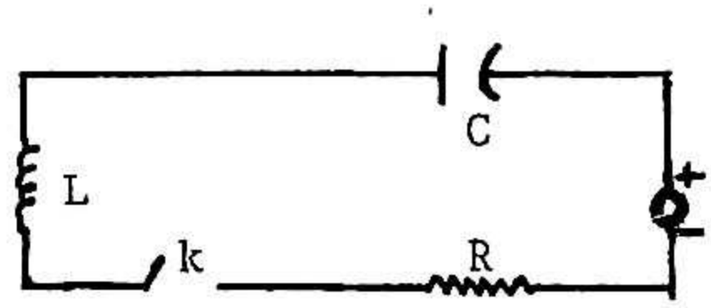
The equation of forced vibration on a vertical spring is

$$mx'' + bx' + kx = A\cos wt, \quad t \geq 0$$

where x is the displacement of the mass from its equilibrium position and m,b,k,A and w are positive constants. The equation has a steady state solution given as

$$x = \frac{A}{\sqrt{(mw^2 - k)^2 + b^2w^2}} \cos(wt + \phi).$$

Use the solution of the spring mass problem given above to obtain the steady state solution for the charge on the capacitor in the circuit given in the diagram below. L,R. and C are constants.



****

The governing differential equation for the circuit is obtained from Kirchhoff laws as

$$L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{Q}{C} = E_0 \cos wt, \quad t \geq 0.$$

The two systems are analogs of each other. Therefore by comparison we may obtain

$$Q = \frac{E_0}{\sqrt{(Lw^2 - \frac{1}{C})^2 + R^2w^2}} \cos(wt + \phi)$$

Since $e^{\lambda t}$ is not zero, we must have

$$(\lambda - i)(\lambda + i)(\lambda^2 + \frac{k}{m}) = 0 ,$$

The solutions are $\lambda_1 = i$, $\lambda_2 = -i$, $\lambda_3 = i\sqrt{\frac{k}{m}}$, $\lambda_4 = -i\sqrt{\frac{k}{m}}$.

Hence, the general solution to the differential equation is

$$C_1 e^{it} + C_2 e^{-it} + C_3 e^{i\sqrt{k/m}} + C_4 e^{-i\sqrt{k/m}} .$$

The particular solution is the part that is not annihilated by the operator $D^2 + \frac{k}{m}$ in the equation (*). That is

$$\underbrace{C_1 e^{it}}_{\cos t + i\sin t} + \underbrace{C_2 e^{-it}}_{\cos t - i\sin t} = \underbrace{(C_1 + C_2)}_{a} \cos t + \underbrace{i(C_1 - C_2)}_{b} \sin t$$

To determine the constant a, b, we use the equation (*).

$$(D^2 + \frac{k}{m})(a\cos t + b\sin t) = \frac{g}{m}\cos t.$$

$$\underbrace{D^2(a\cos t)}_{-a\cos t} + \frac{k}{m} a\cos t + \underbrace{D^2(b\sin t)}_{-b\sin t} + \frac{k}{m} b\sin t = \frac{g}{m}\cos t$$

$$a\left(\frac{k}{m} - 1\right)\cos t + b\left(\frac{k}{m} - 1\right)\sin t = \frac{g}{m}\cos t.$$

The linear independence of cost and sint implies the equalities:

$$a\left(\frac{k}{m} - 1\right) = \frac{g}{m} \quad ; \quad b\left(\frac{k}{m} - 1\right) = 0$$

Hence

$$a = \frac{g}{k - m} \quad ; \quad b = 0$$

The particular solution is $\frac{g}{k-m}\cos t$. This is the answer (A).

5-60

Given the circuit equation, $L\frac{dI}{dt} + RI = E$, and the fact that when $t = 0$, $I = 0$, find I when $t = 1/5$. It is also given that $R = 15$ ohms, $L = 3$ henries and $E = 100$ volts.

****

The integrating factor is $e^{\int \frac{R}{L} dt} = e^{\frac{R}{L}t}$.

Multiply through by this to obtain

$$\left(e^{\frac{R}{L}t} I\right)' = \frac{E}{L} e^{\frac{R}{L}t}.$$

Integration gives

$$e^{\frac{R}{L}t} I = \frac{E}{L} \frac{e^{\frac{R}{L}t}}{\frac{R}{L}} + c,$$

or $$I = \frac{E}{R} + c\, e^{-\frac{R}{L}t}.$$

Since $I = 0$ when $t = 0$, $c = -\frac{E}{R}$.

Thus we have $I = \frac{E}{R}\left(1 - e^{-\frac{R}{L}t}\right)$.

Let $t = 1/5$, $R = 15$, $L = 3$, and $E = 100$ to obtain $I = \frac{100}{15}\left(1 - e^{-1}\right) = 4.21$ amp.

5-59

In a simple RCL series circuit with R = 10 ohms, C = 0.0125 farad, and L = 0.2 henry and the impressed voltage is zero. Find the current in the circuit at any time t if the initial current (at t=0) is zero and the initial charge on the capacitor is 0.5 coulomb.

****

The equation governing the current flow is $\frac{d^2Q}{dt^2} + \frac{R}{L}\frac{dQ}{dt} + \frac{Q}{LC} = 0$, Q is the charge, the current $I = \frac{dQ}{dt}$

The differential equation in Q is of second order with constant coefficients.

Therefore put $Q = e^{rt}$

The characteristic equation is

$$r^2 + \frac{R}{L}r + \frac{1}{LC} = 0$$

$$r = \frac{-R/L \pm \sqrt{R^2/L^2 - 4/LC}}{2}$$

$$= \frac{-10/1/5 \pm \sqrt{100/1/25 - 4/(1/5)(1/80)}}{2}$$

$$= \frac{-50 \pm 30}{2} = -40, -10$$

Therefore $Q(t) = A e^{-40t} + B e^{-10t}$

$$I(t) = \frac{dQ(t)}{dt} = -40A e^{-40t} - 10B e^{-10t}$$

Applying the initial conditions, at $t=0$, $Q(0) = 1/2$, $I(0) = 0$ we get

$$A + B = 1/2$$

$$-40A - 10B = 0$$

Solving $A = -1/6$, $B = 2/3$

Therefore $I(t) = -40(-1/6)e^{-40t} - 10(2/3)e^{-10t}$

$$= \frac{20}{3}(e^{-40t} - e^{-10t}).$$

5-62

An electric circuit contains an 8 ohm resistor in series with an inductor of 0.5 henries and a battery of E volts. At time t = 0 the current is zero. Find the current at any time t > 0 and the maximum current if

(a) E = 64

(b) $E = 32\, e^{-8t}$.

****

The governing differential equation for the current $i(t)$

$$L\frac{di}{dt} + Ri = E(t)$$

where L is the inductance and R is the resistance.

(a)

$$0.5\frac{di}{dt} + 8i = 64 \qquad , i(0) = 0$$

which can be written as

$$\frac{d}{dt}\left(e^{16t}\, i(t)\right) = 128\, e^{16t}.$$

Solving

$$i(t) = 8\left(1 - e^{-16t}\right).$$

The maximum current is obviously equal to 8.

(b)

$$0.5\frac{di}{dt} + 8i = 32e^{-8t} \qquad , i(0) = 0$$

Which can be written as

$$\frac{d}{dt}\left(e^{16t}\, i(t)\right) = 64\, e^{8t}$$

The solution is

$$i(t) = 8\left(e^{-8t} - e^{-16t}\right)$$

To find the maximum current, set $\frac{di}{dt} = 0$ giving $t = \frac{\ln 2}{8}$.

Hence the maximum current $= 8\left(e^{-\ln 2} - e^{-\ln 4}\right)$

$= 2$.

5-61

An R-L circuit has resistance of 10 ohms, inductance of 1 henry and mpressed voltage of 20 volts. The initial current is 0.
a) Find an expression for the current at any time.
b) Find the current at t = 1.
c) When does the current equal 1.5?
d) What is the steady state current? The transient current?

****

a) $R = 10,\ L = 1,\ E(t) = 20 \qquad i(0) = 0$

$10\,i + 1\frac{di}{dt} = 20$. Solve by integrating factor method: $\mu = e^{\int 10\,dt} = e^{10t}$

$$i\,e^{10t} = \int 20e^{10t}\,dt$$

$$ie^{10t} = 2e^{10t} + C$$

$$i = 2 + ce^{-10t}$$

$i(0) = 0 = 2 + C \Rightarrow c = -2$

$$\underline{\underline{i = 2 - 2e^{-10t}}}$$

b) $i(1) = 2 - 2e^{-10} = \underline{\underline{1.999}}$

c) $1.5 = 2 - 2e^{-10t} \Rightarrow -.5 = -2e^{-10t}$

$\Rightarrow e^{-10t} = \frac{-.5}{-2} = .25$

$\Rightarrow -10t = \ln(.25) \Rightarrow t = \frac{\ln(.25)}{-10} = \underline{\underline{.1386}}$ s.

d) $\lim_{t\to\infty}(2 - 2e^{-10t}) = 2$

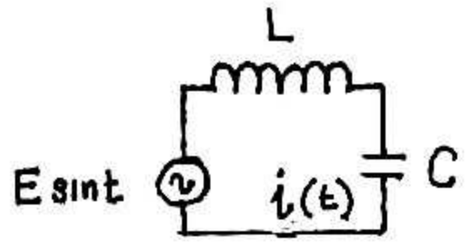
steady state current = $\underline{\underline{2}}$

transient current = $\underline{\underline{-2e^{-10t}}}$

5-64

$$L\frac{di}{dt} + \frac{1}{C}\cdot\int_0^t i(u)du = E\sin t \quad ; \quad i(0) = 0.$$ L, C, E, are constants,

describes the evolution of current i(t) in the following circuit



where L is the inductance and C is the capacitance in the circuit.

$i(t) =$ (A) $(EC/[LC-1])[\sin t - \cos(1/LC)^{\frac{1}{2}}t\,]$

(B) $(EC/[1-LC])[\cos t - \cos(1/LC)^{\frac{1}{2}}t\,]$

(C) $(EC/[1-LC])[\cos t - \sin(1/LC)^{\frac{1}{2}}t\,]$

(D) $(EC/[LC-1])[\sin t - \sin(1/LC)^{\frac{1}{2}}t\,]$

(E) $(EC/[LC-1])[\sin t - e^{-t}\cos(1/LC)^{\frac{1}{2}}t\,]$

****

Dividing by L,
$$(*)\quad \frac{di}{dt} + \frac{1}{LC}\int_0^t i(u)\,du = \frac{E}{L}\sin t$$

The derivative $\frac{di}{dt}$ transforms by

$$\frac{di}{dt} \longrightarrow s\,I(s) - \underbrace{i(0)}_{\text{in } 0} = s\,I(s).$$

The integral transforms by the convolution theorem.

$$\int_0^t 1(u)\,du = 1 * 1 \longrightarrow I(s)\,\frac{1}{s}.$$

Thus, the equation (*) transforms into

$$s\,I + \frac{1}{LC}\,\frac{1}{s}I = \frac{E}{L}\,\frac{1}{s^2+1}$$

Multiplying both sides by s and factoring I,

$$I\left(s^2 + \frac{1}{LC}\right) = \frac{E}{L}\,\frac{s}{s^2+1}$$

5-63

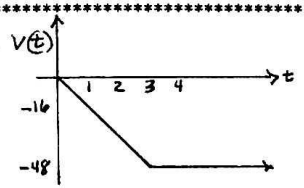
If the applied voltage $V(t) = \begin{cases} -16t & 0 \le t \le 3 \\ -48 & 3 < t \end{cases}$

(a) Graph V(t); find $V'(t)$

(b) Solve: $Li''(t) + \frac{1}{C} i(t) = V'(t)$ ($' = \frac{d}{dt}$); when $L = 1$, $C = \frac{1}{16}$.

****

(a) $y = V(t)$



$$V'(t) = \begin{cases} -16 & 0 < t < 3 \\ 0 & 3 < t \end{cases}$$

If $u_c(t) = \begin{cases} 0 & t < c \\ 1 & t \ge c \end{cases}$

$\Rightarrow V(t) = -16 + 16u_3(t)$

(b) The d.e. becomes $i'' + 16i = -16 + 16u_3(t)$

let $y = i(t) \rightarrow \mathcal{L}\{y\} = Y(s)$

$\mathcal{L}\{y''\} + 16\mathcal{L}\{y\} = -16\mathcal{L}\{1\} + 16\mathcal{L}\{u_3\}$

$$s^2Y(s) - sy(0) - y'(0) + 16Y(s) = -\frac{16}{s} + \frac{16e^{-3s}}{s}$$

$$\Rightarrow [s^2+16]Y(s) = y(0)\cdot s + y'(0) - \frac{16}{s} + 16\frac{e^{-3s}}{s}$$

$$\Rightarrow Y(s) = y(0)\frac{s}{s^2+16} + y'(0)\cdot\frac{1}{s^2+16} - 16\cdot\frac{1}{s(s^2+16)} + 16e^{-3s}\cdot\frac{1}{s(s^2+16)}$$

Note:

$$\frac{1}{s}\cdot\frac{1}{s^2+16} = \frac{A}{s} + \frac{Bs+C}{s^2+16}$$

$1 = A(s^2+16) + Bs^2 + Cs$

s^0: $1 = 16A \Rightarrow A = \frac{1}{16}$

s^1: $0 = C$

s^2: $0 = A + B \Rightarrow B = -\frac{1}{16}$

$$\therefore \frac{1}{s(s^2+16)} = \frac{1}{16}\cdot\frac{1}{s} - \frac{1}{16}\cdot\frac{s}{s^2+16}$$

$\therefore y = i(t) = \mathcal{L}^{-1}\{Y(s)\}$

$$= y(0)\mathcal{L}^{-1}\left\{\frac{s}{s^2+4^2}\right\} + \frac{y'(0)}{4}\mathcal{L}^{-1}\left\{\frac{4}{s^2+4^2}\right\} - 16\mathcal{L}^{-1}\left\{\frac{1}{16}\cdot\frac{1}{s}\right\}$$

$$-16\mathcal{L}^{-1}\left\{-\frac{1}{16}\frac{s}{s^2+16}\right\} + 16\mathcal{L}^{-1}\left\{e^{-3s}\cdot\frac{1}{16}\cdot\frac{1}{s}\right\}$$

$$+16\mathcal{L}^{-1}\left\{e^{-3s}\left(-\frac{1}{16}\right)\frac{s}{s^2+16}\right\}$$

$\therefore y(t) = y(0)\cos 4t + \frac{y'(0)}{4}\sin 4t - \mathcal{L}^{-1}\left\{\frac{1}{s}\right\}$

$$+\mathcal{L}^{-1}\left\{\frac{s}{s^2+16}\right\} + \mathcal{L}^{-1}\left\{e^{-3s}\frac{1}{s}\right\} - \mathcal{L}^{-1}\left\{e^{-3s}\frac{s}{s^2+16}\right\}$$

$$\therefore y = i(t) = y(0)\cos 4t + \frac{y'(0)}{4}\sin 4t - 1 + \cos 4t + u_3(t) - u_3(t)\cos 4(t-3)$$

$$\therefore y = i(t) = \underbrace{[y(0)}_{i(0)} + 1]\cos 4t + \frac{\overbrace{y'(0)}^{i'(0)}}{4}\sin 4t - 1 + u_3(t)[1 - \cos 4(t-3)]$$

MISCELLANEOUS

5-65

A bowl of porridge is served to you at 200°F, too hot to eat. One minute later, it has cooled to 196°. Room temperature is 70°. You can not bear to sit and wait for it to cool to 120°, so you go for a walk. According to Newton's law of cooling, how long should you plan to walk?

****

$u(t)$ = temperature of porridge at time t ; by Newton's law of cooling, $\frac{du}{dt} = -k(u-70)$ and $u(0) = 200$.

The equation $\frac{du}{dt} + ku = 70k$ is linear, with integrating factor $e^{\int k} = e^{kt}$. Then $(e^{kt}u)' = 70ke^{kt}$ and

$e^{kt}u = 70\cancel{k}\frac{e^{kt}}{\cancel{k}} + C$, $\quad u(t) = 70 + Ce^{-kt}$

With $t=0$ and $u=200$, $\quad 200 = 70 + C$, $\quad C = 130$

With $t=1$ and $u=196$, $\quad 196 = 70 + 130e^{-k}$, so $k = -\ln\frac{126}{130} \approx .03$ (we will use $k=.03$ for the rest of the problem, so the answer will only be approximate).

$u(t) = 70 + 130e^{-.03t}$

The temperature is 120° when $120 = 70 + 130e^{-.03t}$

so $t = \frac{\ln\frac{50}{130}}{-.03} \cong 31.85$, the walk should be about

32 minutes long.

$$I(s) = \frac{E}{L}\frac{s}{(s^2+\frac{1}{LC})(s^2+1)} \overset{\text{partial fractions, real numbers}}{=} \frac{E}{L}\left(\frac{As+B}{s^2+\frac{1}{LC}} + \frac{Fs+G}{s^2+1}\right)$$

$$= \frac{E}{L}\left(\frac{(As+B)(s^2+1)+(s^2+\frac{1}{LC})(Fs+G)}{(s^2+\frac{1}{LC})(s^2+1)}\right)$$

Equating numerators, we obtain the equation (*)

$$s = (As+B)(s^2+1)+(s^2+\tfrac{1}{LC})(Fs+G).$$

We find A, B, F, G, by plugging into (*) convenient values for s.

$s^2 = -\frac{1}{LC} \Rightarrow \sqrt{\frac{1}{LC}} = (A\sqrt{\frac{1}{LC}} + B)(-\frac{1}{LC}+1) \Rightarrow A = \frac{1}{1-\frac{1}{LC}} = \frac{LC}{LC-1}$; $B=0$.

(real numbers)

$s^2 = -1 \Rightarrow \sqrt{-1} = (F\sqrt{-1} + G)(-1+\frac{1}{LC}) \Rightarrow F = \frac{1}{-1+\frac{1}{LC}} = \frac{-LC}{LC-1}$; $G=0$.

(real numbers)

In conclusion,

$$I(s) = \frac{E}{L}\left(\frac{\frac{LC}{LC-1}s}{s^2+\frac{1}{LC}} - \frac{\frac{LC}{LC-1}s}{s^2+1}\right) = \frac{E}{L}\frac{LC}{LC-1}\left(\frac{s}{s^2+\frac{1}{LC}} - \frac{s}{s^2+1}\right)$$

$$= \frac{EC}{LC-1}\left(\frac{s}{s^2+\frac{1}{LC}} - \frac{s}{s^2+1}\right)$$

Transforming back to t, $\frac{s}{s^2+\frac{1}{LC}} \to \cos\frac{1}{\sqrt{LC}}t$.

and we obtain,

$$i(t) = \frac{EC}{LC-1}\left(\cos\frac{1}{\sqrt{LC}}t - \cos t\right).$$

$$= \frac{EC}{1-LC}\left(\cos t - \cos\frac{1}{\sqrt{LC}}t\right)$$

The correct answer is (B).

5-67

A baseball pitcher is on the x-axis at x = a (positive). At time t = 0 he throws a ball at speed v toward a batter at the origin. When the ball reaches the origin, the batter hits it back toward the pitcher, imparting to the ball a very short-duration but large-amplitude impulse force of magnitude F. Ignore air resistance and assume the ball stays in a horizontal plane. Using Laplace Transforms, find a function x(t) which describes the position of the ball at any time $t \geq 0$.

****

With initial conditions $x(0) = a$ and velocity $x'(0) = -v$, the ball reaches the origin at time $t_0 = a/v$. The impulse force given to the ball by the batter can be expressed by the Dirac delta-function $F\delta(t-t_0)$. Applying Newton's Second Law of Motion, we have $mx'' = F\delta(t-t_0)$.

Taking the Laplace transform of both sides, $s^2\bar{x} - sa + v = \frac{F}{m}e^{-t_0 s}$

Hence $\bar{x} = \frac{a}{s} - \frac{v}{s^2} + \frac{F}{m}\cdot\frac{e^{-t_0 s}}{s^2}$

Inverting term-by-term, we have

$x(t) = a - vt + \frac{F}{m}(t-t_0)u(t-t_0)$,

where $u(t-t_0)$ denotes the unit step-function which is zero for $t < t_0$ and one for $t \geq t_0$.

5-66

Suppose that a sum of money S_0 is deposited in a bank that pays interest at a rate of 6 percent per annum. Assume the interest is compounded continuously. Find the interest obtained after 5 years. How is it compared with the interest which is compounded four times a year? This means the interest rate is 0.06/4 per quarter and there are 20 quarters.

****

Let $S(t)$ be the sum accumulated after t years. Since the interest is compounded continuously

$$\frac{dS(t)}{dt} = 0.06\, S(t)$$

with initial condition: $S(0) = S_0$. The solution to the above differential equation

$$S(t) = S_0 e^{0.06t}$$

Interest paid after 5 years $= S(5) - S_0$

$$= S_0 (e^{0.3} - 1)$$

$$= 0.34995 S_0.$$

If the interest were compounded four times a year, then

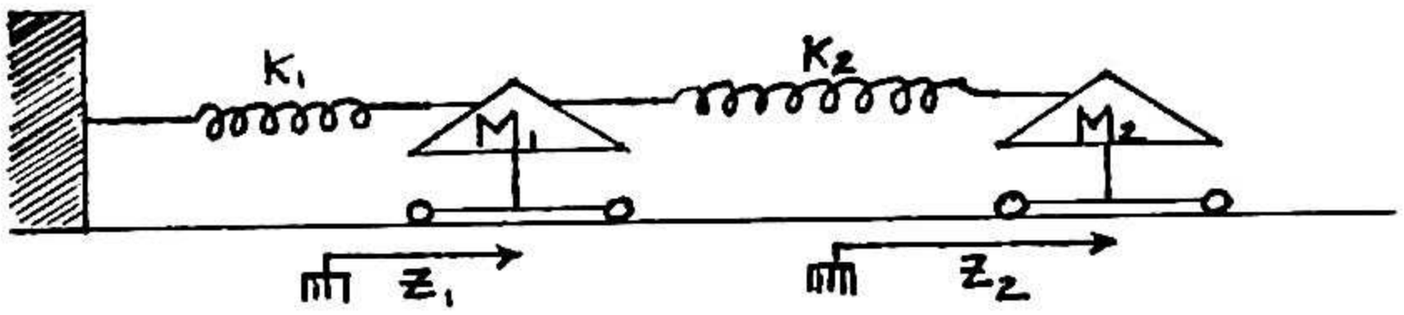
$$S = S_0 \left(1 + \frac{0.06}{4}\right)^{20} = 1.3469 S_0.$$

Interest paid $= S - S_0 = 0.3469 S_0$ which is not much different (less than 1%) as compared with that compounded continuously.

5-69

Two springs are attached to two moveable weights as indicated. The simple springs satisfy Hook's Law (F = Kx) and their spring constants are $K_1 = 5$, $K_2 = 6$, respectively, and their masses are $M_1 = M_2 = 1$. The floor is frictionless and the system is at equilibrium initially. Mass two is given a push to the right which satisfies $Z_2'(0) = 1$. (Where $Z_1(t)$ and $Z_2(t)$ measure the displacement of mass one and two, respectively, from equilibrium.) Find the equations of motion for the masses.

(i.e., find $Z_1(t)$, $Z_2(t)$)



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M_1: $\Sigma F = ma$ $\quad \left[\dot{} = \frac{d}{dt},\ \ddot{} = \frac{d^2}{dt^2}\right]$ M_2: $\Sigma F = ma$

$-K_1 z_1 + K_2(z_2 - z_1) = M_1 \ddot{z}_1$ $\qquad -K_2(z_2 - z_1) = M_2 \ddot{z}_2$

or $(-K_1 - K_2) z_1 + K_2 z_2 = M_1 \frac{d^2 z_1}{dt^2}$ (1) $\qquad$ or $K_2 z_1 - K_2 z_2 = M_2 \frac{d^2 z_2}{dt^2}$ (2)

$K_1 = 5$, $K_2 = 6$, $M_1 = M_2 = 1$, hence (1) and (2) become $*$:

With Initial Conditions:

$z_1(0) = 0$, $z_1'(0) = 0$ [i.e. Centered and at rest]

$z_2(0) = 0$, $z_2'(0) = 1$ [Centered but given a 'push' to the right]

$$* \begin{cases} \frac{d^2 z_1}{dt^2} = -11 z_1 + 6 z_2 \\ \frac{d^2 z_2}{dt^2} = 6 z_1 - 6 z_2 \end{cases}$$

Hence, letting $D = \frac{d}{dt}$, $D^2 = \frac{d^2}{dt^2}$, etc... $*\Rightarrow$:

$$\begin{cases} (D^2+11) z_1 - 6 z_2 = 0 \rightarrow D^2+6: & (D^2+6)(D^2+11) z_1 - 6(D^2+6) z_2 = 0 \\ 6 z_1 - (D^2+6) z_2 = 0 \rightarrow 6: & 36 z_1 - 6(D^2+6) z_2 = 0 \end{cases}$$

Subtracting $\Rightarrow \left[(D^2+6)(D^2+11) - 36\right] z_1 = 0$

or $\left[D^4 + 17D^2 + 30\right] z_1 = 0$

Hence, the characteristic equation is given by:

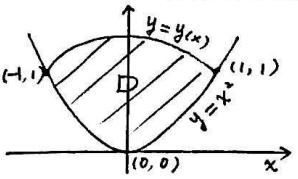
$r^4 + 17r^2 + 30 = 0$

$(r^2+15)(r^2+2) = 0 \Rightarrow r = \pm i\sqrt{2}$, $r = \pm i\sqrt{15}$; Hence,

5-68

Let $y=y(x)$ be a curve with length $\sqrt{2}\pi/2$ on xy plane joining two points $(-1,1)$ and $(1,1)$, such that the domain D enclosed by $y=y(x)$ and the parabola $y=x^2$ has the maximal area. Find the shape of the curve $y=y(x)$.

****



As shown in the figure, the area of the domain D is the integral $\int_{-1}^{1} [y(x) - x^2]\,dx$, while the length of the curve is $\int_{-1}^{1} \sqrt{1+[y'(x)]^2}\,dx$. Now our problem can be restated as the variational problem: To maximize the integral $\int_{-1}^{1} (y(x)-x^2)\,dx$ with the side condition $\int_{-1}^{1} \sqrt{1+(y')^2}\,dx = \frac{\sqrt{2}\pi}{2}$. (1)

Let $F(x, y, y') = (y-x^2) + \lambda\sqrt{1+(y')^2}$, then if $y(x)$ is a solution of (1), then $\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) - \frac{\partial F}{\partial y} = 0$, i.e.

$$\frac{d}{dx}\left(\frac{\lambda y'}{\sqrt{1+(y')^2}}\right) - 1 = 0$$

Integrate both side with respect to x; $\frac{y'}{\sqrt{1+y'^2}} = \frac{x-c_1}{\lambda}$.

Solve this for y': $y' = \frac{(x-c_1)/\lambda}{\sqrt{1-\left(\frac{x-c_1}{\lambda}\right)^2}}$. This is a separable equation, integrate again to get

$$(x-c_1)^2 + (y-c_2)^2 = \lambda^2$$

Hence the shape of the curve $y=y(x)$ is a circle. One can find c_1, c_2 and λ by the conditions that $y(x)$ passes two points $(-1,1)$ and $(1,1)$ and $\int_{-1}^{1} \sqrt{1+(y')^2}\,dx = \frac{\sqrt{2}\pi}{2}$.

5-70

The separable DE

$$\frac{dr}{d\theta} = \frac{1}{2} r^2 \sin\theta$$

arises in Celestial Mechanics. The trajectories it describes are predicted by Kepler's Laws.

Find the solution such that r = 1 when θ = 0. Show that as θ approaches a certain (finite) value, r becomes infinite.

****

Since the equation is separable,

$$\frac{2dr}{r^2} = \sin\theta\, d\theta$$

Integrate $\int \frac{2dr}{r^2} = \int \sin\theta\, d\theta$

$$\Rightarrow \quad -\frac{2}{r} = -\cos\theta + C$$

Since $r = 1$, when $\theta = 0 \Rightarrow \quad C = -1$

Thus $\boxed{r = \frac{2}{1+\cos\theta}}$

As $\theta \uparrow \pi$, $r \to \infty$. This is a parabolic trajectory.

the solution for z_1 is given by:

$z_1(t) = c_1 \sin\sqrt{2}t + c_2 \cos\sqrt{2}t + c_3 \sin\sqrt{15}t + c_4 \cos\sqrt{15}t$

then & $z_2 = (D^2+11)\, z_1$

$\Rightarrow z_2 = 1.5c_1 \sin\sqrt{2}t + 1.5c_2 \cos\sqrt{2}t - \frac{2}{3}c_3 \sin\sqrt{15}t - \frac{2}{3}c_4 \cos\sqrt{15}t$

using the initial conditions:

$0 = z_1(0) = c_2 + c_4$

$0 = z_1'(0) = \sqrt{2}c_1 + \sqrt{15}c_3$

$0 = z_2(0) = 1.5c_2 - \frac{2}{3}c_4$

$1 = z_2'(0) = 1.5\sqrt{2}c_1 - \frac{2}{3}\sqrt{15}c_3$

$\Rightarrow c_2 = c_4 = 0$

$\Rightarrow \sqrt{2}c_1 = -\sqrt{15}c_3$

$\Rightarrow c_1 = -\frac{\sqrt{15}}{\sqrt{2}}c_3$

sub in:

$1 = 1.5\sqrt{2}\left(-\frac{\sqrt{15}}{\sqrt{2}}\right)c_3 - \frac{2}{3}\sqrt{15}c_3$

$1 = -1.5\sqrt{15}c_3 - \frac{2}{3}\sqrt{15}c_3$

$1 = \sqrt{15}c_3\left[-1.5 - \frac{2}{3}\right]$

$1 = -\frac{13}{6}\sqrt{15}c_3 \Rightarrow c_3 = \frac{-6}{13\sqrt{15}} \Rightarrow \frac{6}{13\sqrt{2}} = c_1$

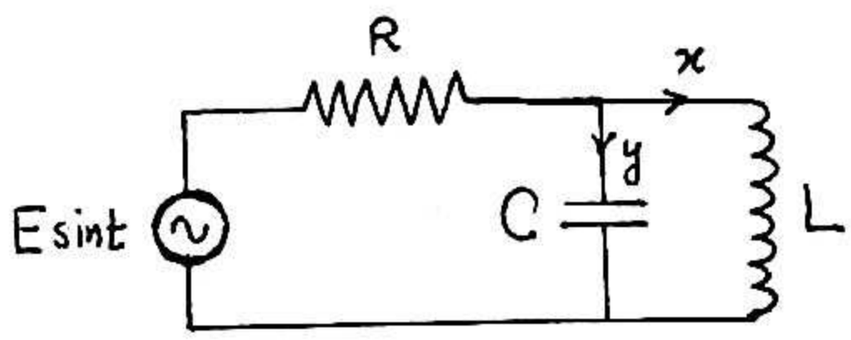
or $c_3 = \frac{-2\sqrt{15}}{65} \Rightarrow$ or $\frac{3\sqrt{2}}{13} = c_1$

$\therefore z_1(t) = \frac{3\sqrt{2}}{13}\sin\sqrt{2}t - \frac{2\sqrt{15}}{65}\sin\sqrt{15}t$

and $z_2(t) = \frac{9}{26}\sqrt{2}\sin\sqrt{2}t + \frac{4\sqrt{15}}{195}\sin\sqrt{15}t$

5-72

The evolution of the currents x(t) and y(t) in the given circuit is described by the following equations



$$L\frac{dx}{dt} + R(x + y) = E\sin t$$

$$L\frac{dx}{dt} + \frac{1}{C}\int_0^t y(u)du = 0 \qquad x(0) = y(0) = 0. \quad E, R, L, C \text{ are constants.}$$

$$X(s) = \text{(A)} \quad -\frac{E}{RLC}\left[\, 1/(s^2+1)(s^2 - \frac{1}{LC}\, s - \frac{1}{RC})\right]$$

$$\text{(B)} \quad -\frac{E}{RLC}\left[\, 1/(s^2+1)(s^2 - \frac{1}{RC}\, s - \frac{1}{LC})\right]$$

$$\text{(C)} \quad -\frac{E}{RLC}\left[\, s/(s^2+1)(s^2 - \frac{1}{LC}\, s - \frac{1}{RC})\right]$$

$$\text{(D)} \quad -\frac{E}{RLC}\left[\, s/(s^2+1)(s^2 - \frac{1}{RC}\, s - \frac{1}{LC})\right]$$

(E) None

****

Dividing both equations by L

$$\frac{dx}{dt} + \frac{R}{L}x + \frac{R}{L}y = \frac{E}{L}\sin t$$

$$\frac{dx}{dt} + \frac{1}{LC}\int_0^t y(u)\,du = 0.$$

The equations transform to

$$sX - \underbrace{x(0)}_{0} + \frac{R}{L}X + \frac{R}{L}Y = \frac{E}{L}\frac{1}{s^2+1}$$

$$sX - \underbrace{x(0)}_{0} + \frac{1}{LC}\frac{1}{s}Y = 0$$

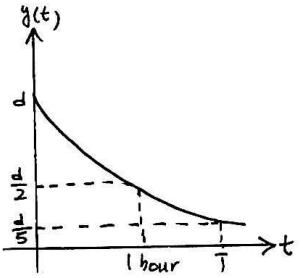
5-71

Suppose that a dose of d mg of a drug is injected into the bloodstream. Assume that the drug leaves the blood and enters the urine at a rate proportional to the amount of the drug present in the blood. Assume in addition that half of the drug dose has entered the urine after 1 hour. Find the time at which the amount of drug in the bloodstream is 20 percent of the original dose d.

****

Let $y(t)$ be the amount of drug in the bloodstream at time t. Since the drug leaves the blood at a rate proportional to $y(t)$, we have

$$\frac{dy}{dt} = -ky \quad \text{for some constant } k$$



and $y(0) = d$. We also know that $y(1) = d/2$.

Solving the differential equation

$$y(t) = y(0)\, e^{-kt} = d\, e^{-kt}.$$

To find k, we use

$$y(1) = \frac{d}{2} = d\, e^{-k} \quad \text{giving} \quad k = \ln 2.$$

To find the time T such that $y(T) = d/5$.

$$y(T) = \frac{d}{5} = d\, e^{-kT}$$

giving $\quad kT = \ln 5.$

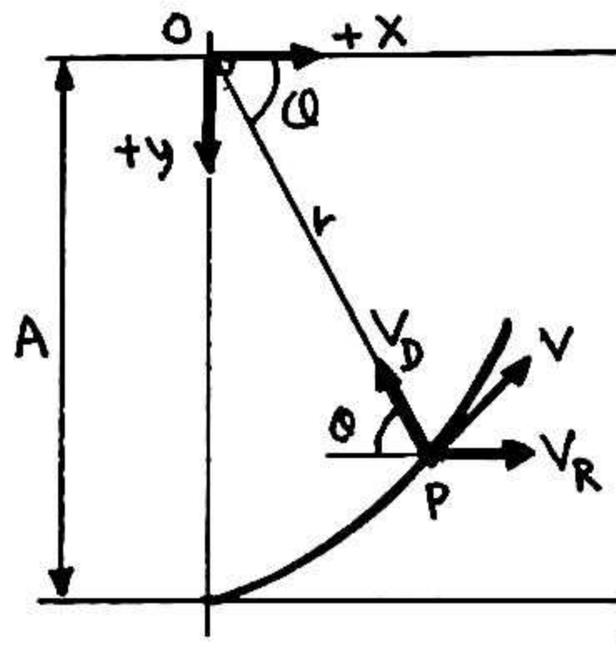
Hence $\quad T = \dfrac{\ln 5}{\ln 2}.$

5-73

As a classic problem in kinematics, a duck attempts to swim across a river of width A. The locus of the duck (that is, its equation of motion) is required if it is assumed that the duck swims across the river in such a manner that it always heads for the point on the bank directly opposite its starting position.

Herein, let V_R = the velocity of the river, V_D = the velocity of the duck in still water, and respectively V_x , V_y = the x component and the y component of the resultant velocity of motion.

****



Referring to the adjacent sketch, the equations of motion of the duck are:

$$\begin{cases} V_x \equiv \dfrac{dx}{dt} = V_R - V_D \cdot \cos\theta \\ V_y \equiv \dfrac{dy}{dt} = -V_D \cdot \sin\theta \end{cases}$$

Thus,

$$\frac{dy}{dx} \equiv \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{-V_D \cdot \sin\theta}{V_R - V_D \cdot \cos\theta} = \frac{-\frac{y}{r}}{K - \left(\frac{x}{r}\right)} \; ;$$

$$K = \frac{V_R}{V_D} \; , \qquad r = \sqrt{x^2+y^2} \; .$$

This equation, $\dfrac{dy}{dx} = \dfrac{y}{x - K \cdot \sqrt{x^2+y^2}}$, is a homogeneous, first order ordinary differential equation. With $w = \frac{y}{x}$, the equation becomes 'separable'; that is, the equation is re-written as $\left(x \cdot \dfrac{dw}{dx} + w\right) = \dfrac{w}{1 - K \cdot \sqrt{1+w^2}}$, yielding

$$\int \frac{dw}{w \cdot \sqrt{1+w^2}} - K \cdot \int \frac{dw}{w} = K \cdot \int \frac{dx}{x} \quad , \text{ and}$$

Factoring X and Y,

$$(I) \qquad \left(s+\frac{R}{L}\right)X + \frac{R}{L}Y = \frac{E}{L}\frac{1}{s^2+1}$$

$$(II) \qquad sX + \frac{1}{LC}\frac{1}{s}Y = 0$$

We eliminate Y by multiplying $(I)\times\left(-\frac{1}{RCs}\right)$ and adding to II.

$$-\frac{1}{RCs}\left(s+\frac{R}{L}\right)X - \frac{1}{LCs}Y = -\frac{1}{RCs}\frac{E}{L}\frac{1}{s^2+1}$$

$$\oplus \qquad sX + \frac{1}{LCs}Y = 0$$

$$-\frac{1}{RCs}\left(s+\frac{R}{L}\right)X + sX = -\frac{1}{RCs}\frac{E}{L}\frac{1}{s^2+1}$$

Multiplying both sides by s, and factoring X,

$$X\left(-\frac{1}{RC}s - \frac{1}{LC} + s^2\right) = -\frac{E}{RLC}\frac{1}{s^2+1}$$

Hence,

$$X(s) = -\frac{E}{RLC}\frac{1}{\left(s^2-\frac{1}{RC}s-\frac{1}{LC}\right)(s^2+1)}.$$

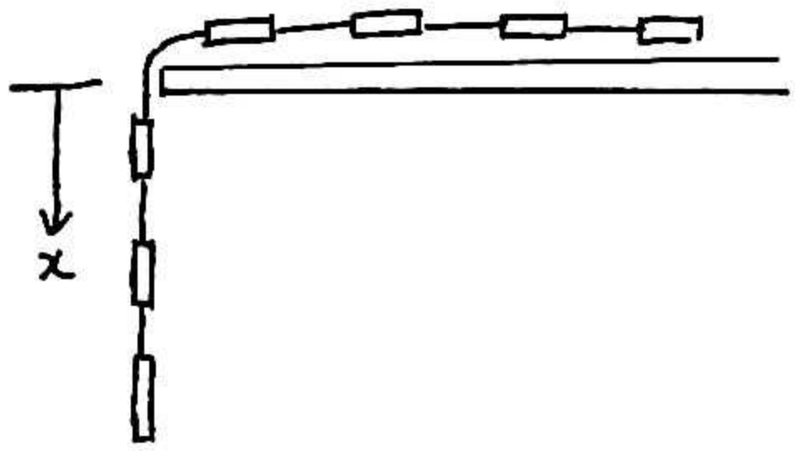
This is the answer (B).

5-74

A uniform chain of length a is placed on a horizontal frictionless table so that a length b of the chain dangles over the side. How long will it take for the chain to slide off the table?

****

Suppose at time t a length x of the chain is dangling over the side. Assume σ to be the mass per unit length of the chain. By Newton's second law



$$\text{net force} = (\sigma g)\, x = \sigma a \frac{dv}{dt}$$

where V(t) is the velocity of the chain.

Using $\frac{dv}{dt} = v\frac{dv}{dx}$, so the governing equation becomes

$$v\frac{dv}{dx} = \frac{gx}{a}.$$

Using separation of variables and integrating

$$v^2 = \frac{g}{a}x^2 + c.$$

The initial conditions are : $t=0$, $x=b$, $v=0$. This gives $c = -\frac{g}{a}b^2$. Next

$$v = \frac{dx}{dt} = \sqrt{\frac{g}{a}}\sqrt{x^2-b^2}.$$

Again seperation of variables and incorporating the initial condition : $X(0)=b$ and $X(T)=a$ where T is the time for the chain to slide off the table.

$$\int_0^a \frac{dx}{\sqrt{x^2-b^2}} = \sqrt{\frac{g}{a}}\int_0^T dt$$

then

$$T = \sqrt{\frac{a}{g}}\ln\left(\frac{a+\sqrt{a^2-b^2}}{b}\right).$$

$$\ln\left(\frac{W}{1+\sqrt{1+W^2}}\right) - K\cdot\ln W = K\cdot\ln X + K\cdot\ln C$$

Via logarithms, conversion back to the xy plane gives

$$y^{1-K} = C^K\cdot\left(X+\sqrt{X^2+y^2}\right).$$

At $X=0, y=A$. Thus $C^K = A^{-K}$. We also find that

$$\left(\frac{y}{A}\right)^{1+K} = \frac{1}{A}\cdot\left(\sqrt{X^2+y^2}-X\right), \text{ and } \left(\frac{y}{A}\right)^{1-K} = \frac{1}{A}\cdot\left(\sqrt{X^2+y^2}+X\right).$$

Subtraction yields the required locus:

$$X = f(y) = \frac{A}{2}\cdot\left[\left(\frac{y}{A}\right)^{1-K} - \left(\frac{y}{A}\right)^{1+K}\right] \quad ; \quad K = \frac{V_R}{V_D}.$$

Case Studies

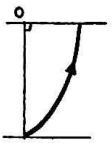
$K=0$ $(V_R=0, V_D\neq 0)$

locus: $X=0$

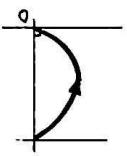
$K=1$ $(V_R=V_D\neq 0)$

locus:

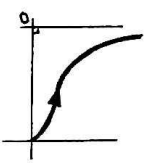
$$y^2 = -2A\cdot\left(X-\frac{A}{2}\right)$$



$0<K<1$ $(V_R<V_D)$



$K>1$ $(V_R>V_D)$



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